NON COMMUTATIVE FINITE DIMENSIONAL MANIFOLDS II:

MODULI SPACE AND STRUCTURE

OF NON COMMUTATIVE 3-SPHERES

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ABSTRACT. This paper contains detailed proofs of our results on the moduli space and the structure of noncommutative 3-spheres. We develop the notion of central quadratic form for quadratic algebras, and a general theory which creates a bridge between noncommutative differential geometry and its purely algebraic counterpart. It allows to construct a morphism from an involutive quadratic algebras to a C*-algebra constructed from the characteristic variety and the hermitian line bundle associated to the central quadratic form. We apply the general theory in the case of noncommutative 3-spheres and show that the above morphism corresponds to a natural ramified covering by a noncommutative 3-dimensional nilmanifold. We then compute the Jacobian of the ramified covering and obtain the answer as the product of a period (of an elliptic integral) by a rational function. We describe the real and complex moduli spaces of noncommutative 3-spheres, relate the real one to root systems and the complex one to the orbits of a birational cubic automorphism of three dimensional projective space. We classify the algebras and establish duality relations between them.

1. Introduction

This paper contains detailed proofs of our results on the moduli space and the structure of noncommutative 3-spheres announced in [15].

Through the analysis of a specific class of noncommutative manifolds which arose as solutions of a simple equation of K-theoretic origin we discovered rather general structures which lie at the intersection of two fundamental aspects of noncommutative geometry, namely

- Differential Geometry
- Algebraic Geometry

This class of noncommutative manifolds, called noncommutative 3-spheres, has a very rich structure both at the level of the objects themselves as well as at the level of the moduli space which parameterizes these geometric objects. There are two aspects in the geometry of the moduli space :

- The *real* moduli space and its scaling foliation, its link with the alcove structure of the root system D_3 and the Morse theory of the character of the signature representation.
- The *complex* moduli space and its net of elliptic curves, its link with the iteration of a cubic transformation of $\mathbb{P}_3(\mathbb{C})$.

At the level of the structure of the noncommutative 3-spheres our main result is to relate them to very well understood noncommutative nilmanifolds which fall under the framework of the early theory developped in [8] and were analysed in great detail in [1] [2].

The core of the paper is to construct the corresponding map of noncommutative spaces and compute its Jacobian. The essence of the work is to extract from very complicated computations the general concepts that allow not only to understand what is going on but also to extend the construction in full generality. Thus at center stage lies the computation of the Jacobian and the gradual simplification of the result which at first was expressed in terms of elliptic functions and the 9-th power of Dedekind η function. We shall reach at the end of the paper (Theorem 13.8) a result of utmost simplicity while the starting point was a computation performed even in the trigonometric case with the help of a computer[1]. While we reach a reasonable level of conceptual understanding of the general construction of the map, we believe that a lot remains to be discovered for the abstract construction of the calculus as well as for the general computation of the Jacobian, let alone in the case of higher dimensional spheres which we do not adress here.

After recalling the basic definitions and properties of the noncommutative 3-spheres in section 2 we analyse the real moduli space in section 3 and exhibit a fundamental domain in terms of alcoves of the root system D_3.

In section 4 we define the scaling foliation and show its compatibility with the alcove structure of the real moduli space. We also show that the isomorphism class of the 4-spaces $\mathbb{R}^4(\Lambda)$ remains constant on the leaves of the foliation. In order to prove the converse *i.e.* that isomorphism of the 4-spaces $\mathbb{R}^4(\Lambda)$ implies equality of the leaves we compute in details in section 5 the geometric data of the quadratic algebra of $\mathbb{R}^4(\Lambda)$. This allows to finish the proof of the converse in section 6.

We exhibit in section 7 more subtle relations between the 4-spaces $\mathbb{R}^4(\Lambda)$ given by *dualities*. At the level of the algebras these are obtained from the general notion of semi-cross product of quadratic algebras. At the level of the moduli space these dualities shrink further the fundamental domain and that amounts essentially to the transition from the root system D_3 to the larger one C_3.

We then use these to describe the 4-spaces $\mathbb{R}^4(\Lambda)$ for degenerate values of the parameter in section 8.

In section 9 we analyse the complex moduli space which appears naturally as a net of elliptic curves having eight points in common in $\mathbb{P}_3(\mathbb{C})$. We show that in the generic case these elliptic curves are the characteristic varieties of the algebras that their points label. Moreover the canonical correspondence σ is simply the restriction of a globally defined cubic map of $\mathbb{P}_3(\mathbb{C})$. This gives in particular a very natural choice of generators for the algebra. As a preparation for the next section we give the natural parameterization of the net of elliptic curves in terms of ϑ-functions.

Section 10 is the most technical one and contains the root of the concepts developped in full generality in section 11. In essence what we do first is starting from unitary representations of the Sklyanin algebra constructed by Sklyanin in his original paper we derive a one parameter family of $\star$-homorphisms from the algebras of $\mathbb{R}^4(\Lambda)$ in the generic case, to the algebras of noncommutative tori. We then use a suitable restriction to the 3-spheres $S^3(\Lambda)$. After a lot of work on this construction we find that we can eliminate all occurences of ϑ-functions from the formulas and obtain a purely algebraic formulation of the construction as a morphism to a twisted cross product C^*-algebra obtained from the geometric data.

Section 11 describes the abstract general construction in the framework of involutive quadratic algebras. The key notions are those of *central quadratic form* and of *positivity* for such forms. It is in that section that the interaction between the two above aspects of noncommutative geometry is manifest. In fact we construct a bridge between the purely algebraic notions such as the geometric data of a quadratic algebra and the world of noncommutative geometry including the topological (C^*-algebraic)

[1]We are grateful to Michael Trott for his help

and differential geometric aspects (in the sense of [8], [9]). At one end of the bridge one starts with the given involutive quadratic algebra. At the other one has the C^*-algebra obtained as the twisted cross product of the charateristic variety by the canonical correspondence. The twisting is effected by an hermitian line bundle and the construction is a special case of a general one due to M. Pimsner. The bridge provides a $\star$-algebra morphism. This algebra morphism has a "trivial part" which does not make use of the central quadratic form and lands in a "triangular" subalgebra of the C^*-algebra. This part was well-known for quite sometime to noncommutative algebraic geometors. The non-trivaility of our construction lies in the involved relations coming from the cross terms mixing generators with their adjoints.
We compute in section 12 the Jacobian of the above map. We first define what we mean by the jacobian in the sense of noncommutative geometry where Hochschild homology replaces differential forms. The algebraic form of the result then suggests the existence of a calculus of purely algebraic nature allowing to express the cyclic cohomology fundamental class in terms of the algebraic geometry of the characteristic variety and a hermitian structure on the canonical line bundle. This is achieved in section 13 and allows to finally obtain the purest form of the computation of the jacobian in the already mentionned Theorem 13.8. Needless to say this is a paper of highly "computational" nature and we tried to ease the reading by supplying in an appendix the basic factorisations of the minors and the sixteen theta relations which are often used in the text.

Contents

2. The noncommutative 3-spheres $S^3(\Lambda) \subset \mathbb{R}^4(\Lambda)$

We shall recall in this section the basic properties of the noncommutative spheres $S^3(\Lambda)$ and the corresponding 4-spaces $\mathbb{R}^4(\Lambda)$.

2.1. **Unitary "up to scale".**

We let $\mathcal{A}$ be a unital involutive algebra and first start with a unitary "up to scale" in $M_q(\mathcal{A})$, *i.e.*

$$U \in M_q(\mathcal{A}), \quad UU^* = U^*U \in \mathcal{A} \otimes \mathbb{1} \subset \mathcal{A} \otimes M_q(\mathbb{C}). \tag{2.1}$$

Lemma 2.1. *Let $U \in M_q(\mathcal{A})$ satisfy (2.1) with*

$$UU^* = U^*U = C \otimes \mathbb{1} \in \mathcal{A} \otimes M_q(\mathbb{C})$$

then C is in the center of the $*$-algebra generated by the matrix elements of U.

Proof. One has $(C\otimes\mathbb{1})U = (UU^*)U = U(U^*U) = U(C\otimes\mathbb{1})$ and $(C\otimes\mathbb{1})U^* = (U^*U)U^* = U^*(UU^*) = U^*(C\otimes\mathbb{1})$ by associativity in $M_q(\mathcal{A})$, which implies the result. □

Let

$$\tau_\mu, \ \mu \in \{0, \dots, q^2 - 1\} \tag{2.2}$$

be an orthonormal basis of $M_q(\mathbb{C})$ for the scalar product $\langle A|B\rangle = \frac{1}{q}\mathrm{Trace}(A^*B)$. Then one has

$$U = \tau_\mu z^\mu, \quad z^\lambda \in \mathcal{A} \tag{2.3}$$

where we used the Einstein summation convention on "up-down indices". The $*$-algebra generated by the matrix elements of U is the $*$-subalgebra of $\mathcal{A}$ generated by the $z^\lambda, z^{\mu*}$ ($\lambda, \mu \in \{0, \dots, q^2-1\}$) and if U is as in the above lemma then one has the equalities

$$C = \sum_\mu z^\mu z^{\mu*} = \sum_\mu z^{\mu*} z^\mu \tag{2.4}$$

for the central element C of the $*$-algebra generated by the z^μ.

2.2. **Equation** $\mathrm{ch}_{1/2}(U) = 0$.

We now turn to the relation $\mathrm{ch}_{1/2}(U) = 0$, in the unreduced complex *i.e.* using the convention of summation on repeated indices,

$$U^{i_0}_{i_1} \otimes (U^*)^{i_1}_{i_0} - (U^*)^{i_0}_{i_1} \otimes U^{i_1}_{i_0} = 0 \tag{2.5}$$

where the left hand side belongs to the tensor square $\mathcal{A}^{\otimes 2} = \mathcal{A} \otimes \mathcal{A}$. Both terms in the left hand side give sums of the q^2 terms of the form $(z^\mu \otimes z^{\nu*})\,(\tau_\mu)^{i_0}_{i_1}\,(\tau^*_\nu)^{i_1}_{i_0}$ and similarly for the other. The sum on i_0, i_1 is thus simply Trace $(\tau_\mu\,\tau^*_\nu)$, *i.e.* the Hilbert Schmidt inner product $(\tau_\mu, \tau_\nu) = q\langle\tau_\nu|\tau_\mu\rangle$. This is 0 unless $\mu = \nu$ and is q if $\mu = \nu$. Thus, up to an overall factor of q the equality (2.5) means:

$$\sum (z^\mu \otimes z^{\mu*} - z^{\mu*} \otimes z^\mu) = 0\,. \tag{2.6}$$

Lemma 2.2. *Equation* (2.6) *holds iff there exists a unitary symmetric matrix $\Lambda \in M_{q^2}(\mathbb{C})$ such that:*

$$z^{\mu*} = \Lambda^\mu_\nu\, z^\nu\,. \tag{2.7}$$

Proof. Let us first assume that the z^μ are linearly independent elements of $\mathcal{A}$. Let then φ_μ be linear forms on $\mathcal{A}$ with $\varphi_\mu(z^\nu) = \delta^\nu_\mu$. Applying $1 \otimes \varphi_\mu$ to (2.6) we get

$$z^{\mu *} = \sum z^\nu \, \varphi_\mu(z^{\nu *}) = \sum \Lambda^\mu_\nu \, z^\nu$$

where the matrix Λ is uniquely prescribed by this relation. Then since the $z^\mu \otimes z^\nu$ are linearly independent in $\mathcal{A} \otimes \mathcal{A}$ the relation (2.6) means, looking at the coefficient of $z^\mu \otimes z^\nu$ on both sides,

$$\Lambda^\mu_\nu = \Lambda^\nu_\mu \tag{2.8}$$

so that the matrix Λ is symmetric.

Taking the adjoint of both sides in $z^{\mu *} = \Lambda^\mu_\nu \, z^\nu$ one gets

$$z^\mu = \bar{\Lambda}^\mu_\nu \, z^{\nu *} = \bar{\Lambda}^\mu_\nu \, \Lambda^\nu_\rho \, z^\rho = (\Lambda^* \Lambda)^\mu_\rho \, z^\rho$$

and the linear independence of the z^ρ thus shows that:

$$\Lambda^* \Lambda = 1 \, . \tag{2.9}$$

For the general case[2] note that equation (2.6) is invariant by the transformation

$$\tilde{z}^\mu = W^\mu_\nu \, z^\nu \, .$$

where $W \in U(q^2)$ is a unitary matrix. Moreover (2.7) implies a similar equation for $(\tilde{z}^\mu)$ with the matrix

$$\tilde{\Lambda} = \bar{W} \, \Lambda \, \bar{W}^t \, ,$$

which is still symmetric and unitary. This allows to assume that the kernel of the linear map from $\mathbb{C}^{q^2}$ to $\mathcal{A}$ determined by $e^\nu \mapsto z^\nu$ is the linear span of a subset I of the basis vectors e^ν. In other words the non-zero z^ν are linearly independent and the above proof ensures the existence of a matrix fulfilling (2.7) which is symmetric and unitary once extended by the identity on the e^ν , $\nu \in I$.

□

As pointed out in [13] and as will be explained in Part III, for the study of $(2n+1)$-dimensional spherical manifolds one should take $q = 2^n$ to be coherent in particular with the suspension functor. In the following we shall concentrate on the 3-dimensional case (and the corresponding noncommutative $\mathbb{R}^4$) which is the lowest dimensional non trivial case from the noncommutative side and for which $\mathrm{ch}_{1/2}(U) = 0$ is the only K-homological condition. Accordingly we take $q = 2$ in the following.

2.3. **Noncommutative 3-spheres and 4-planes.**

We now turn to the case $q = 2$ and we choose as orthonormal basis of $M_2(\mathbb{C})$ the basis

$$\tau_0 = \mathbb{1} \text{ and } \tau_k = i\sigma_k, \;\; k \in \{1, 2, 3\} \tag{2.10}$$

where the σ_k are the Pauli matrices, *i.e.*

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.11}$$

which satisfy $\sigma_j^* = \sigma_j$, $\sigma_j^2 = 1$ and $\sigma_k \, \sigma_\ell = i \, \varepsilon_{k\ell m} \, \sigma_m$ for any permutation (k, ℓ, m) of (1, 2, 3), where $\varepsilon_{123} = 1$ and ε is totally antisymmetric. This allows to write

$$U = z^0 + i \, \sigma_1 \, z^1 + i \, \sigma_2 \, z^2 + i \, \sigma_3 \, z^3, \qquad z^\mu \in \mathcal{A} \tag{2.12}$$

and one has $U^* = z^{0*} - \, i \, \sigma_1 \, z^{1*} - \, i \, \sigma_2 \, z^{2*} - \, i \, \sigma_3 \, z^{3*}$.

[2] The simplification of the argument of [13] given here is due to G. Skandalis

Thus

$$\begin{aligned} UU^* &= z^0 z^{0*} + z^1 z^{1*} + z^2 z^{2*} + z^3 z^{3*} \\ &+ i\,\sigma_1\,(z^1 z^{0*} - z^0 z^{1*} + z^2 z^{3*} - z^3 z^{2*}) \\ &+ i\,\sigma_2\,(z^2 z^{0*} - z^0 z^{2*} + z^3 z^{1*} - z^1 z^{3*}) \\ &+ i\,\sigma_3\,(z^3 z^{0*} - z^0 z^{3*} + z^1 z^{2*} - z^2 z^{1*})\,. \end{aligned}$$

Similarly we get,

$$\begin{aligned} U^*U &= z^{0*} z^0 + z^{1*} z^1 + z^{2*} z^2 + z^{3*} z^3 \\ &+ i\,\sigma_1\,(z^{0*} z^1 - z^{1*} z^0 + z^{2*} z^3 - z^{3*} z^2) \\ &+ i\,\sigma_2\,(z^{0*} z^2 - z^{2*} z^0 + z^{3*} z^1 - z^{1*} z^3) \\ &+ i\,\sigma_3\,(z^{0*} z^3 - z^{3*} z^0 + z^{1*} z^2 - z^{2*} z^1)\,. \end{aligned}$$

Thus equation (2.1) is equivalent to 7 relations which are,

$$\sum z^\mu z^{\mu*} = \sum z^{\mu*} z^\mu \tag{2.13}$$

$$z^k z^{0*} - z^0 z^{k*} + \sum \varepsilon_{k\ell m}\, z^\ell z^{m*} = 0 \tag{2.14}$$

$$z^{0*} z^k - z^{k*} z^0 + \sum \varepsilon_{k\ell m}\, z^{\ell*} z^m = 0\,. \tag{2.15}$$

We then let $\mathcal{S}$ be the space of unitary symmetric matrices,

$$\mathcal{S} = \{\Lambda \in M_4(\mathbb{C})\,;\ \Lambda\Lambda^* = \Lambda^*\Lambda = 1\,,\ \Lambda^t = \Lambda\}\,. \tag{2.16}$$

We define for $\Lambda \in \mathcal{S}$ the algebra $C_{\rm alg}(\mathbb{R}^4(\Lambda))$ of coordinates on $\mathbb{R}^4(\Lambda)$ as the algebra generated by the z^μ, $z^{\mu*}$ with the relations (2.14), (2.15) together with,

$$z^{\mu*} = \Lambda^\mu_\nu\, z^\nu \tag{2.17}$$

Note that (2.13) follows automatically from (2.17). For $S^3(\Lambda)$ we add the inhomogeneous relation,

$$\sum z^{\mu*} z^\mu = 1\,. \tag{2.18}$$

By Lemma 2.1 the element $C = \sum z^{\mu*} z^\mu$ is in the center of the involutive algebra $C_{\rm alg}(\mathbb{R}^4(\Lambda))$.

Theorem 2.3. *Let $\mathcal{A}$ be a unital involutive algebra and $U \in M_2(\mathcal{A})$ a unitary such that $\mathrm{ch}_{1/2}(U) = 0$. Then there exists $\Lambda \in \mathcal{S}$ and a homomorphism $\varphi : C_{\rm alg}(S^3(\Lambda)) \to \mathcal{A}$ such that:*

$$U = \varphi(\tau_\mu\, z^\mu)\,.$$

Conversely for any Λ the unitary $U = \tau_\mu\, z^\mu \in M_2(C_{\rm alg}(S^3(\Lambda)))$ fulfills $\mathrm{ch}_{1/2}(U) = 0$.

By construction we thus obtain involutive algebras parametrized by $\Lambda \in \mathcal{S}$. They are endowed with a canonical element of $H_3(\mathcal{A}, \mathcal{A})$ (in fact of Z_3) given by

$$[S^3(\Lambda)] = \mathrm{ch}_{3/2}(U) = U^{i_0}_{i_1} \otimes U^{*i_1}_{i_2} \otimes U^{i_2}_{i_3} \otimes U^{*i_3}_{i_0} - U^{*i_0}_{i_1} \otimes U^{i_1}_{i_2} \otimes U^{*i_2}_{i_3} \otimes U^{i_3}_{i_0}\,. \tag{2.19}$$

The operations $U \to \lambda U$, $U \to V_1 U V_2$, $U \to U^*$, for $\lambda \in U(1)$ and $V_j \in SU(2)$ together with the universality described in Theorem 2.3 give natural isomorphisms between the $S^3(\Lambda)$ as follows,

Proposition 2.4. *The following define isomorphisms $S^3(\Lambda) \to S^3(\Lambda')$ (resp. $\mathbb{R}^4(\Lambda) \to \mathbb{R}^4(\Lambda')$) preserving $[S^3]$ (resp. $[\mathbb{R}^4]$) in the first two cases and changing its sign in the last case:*

1) *For $\lambda \in U(1)$, $\Lambda' = \lambda^2 \Lambda$ one lets*

$$\varphi(z^\mu_{\Lambda'}) = \lambda^{-1}\, z^\mu_\Lambda\,.$$

2) *For $V \in SO(4)$, $\Lambda' = V\Lambda V^t$ one lets*

$$\varphi(z^\mu_{\Lambda'}) = V^\mu_\nu\, z^\nu_\Lambda\,.$$

3) *For $\Lambda' = \varepsilon\,\Lambda^{-1}\,\varepsilon$ one lets*

$$\varphi(z^\mu_{\Lambda'}) = \varepsilon_\mu\, z^{\mu *}_\Lambda\,,\ \varepsilon_0 = 1\,,\ \varepsilon_k = -1\,,\ \varepsilon_{\mu\nu} = 0 \quad \mu \neq \nu\,.$$

Proof. 1) Let $U = U_\Lambda$ be the unitary in $M_2(C_{\rm alg}(S^3(\Lambda))$. Then λU still fulfills $\mathrm{ch}_{1/2}(\lambda U) = 0$ and $\mathrm{ch}_{3/2}(\lambda U) = \mathrm{ch}_{3/2}(U)$. With $\tilde z^\mu = \lambda\, z^\mu$ one has $(\tilde z^\mu)^* = \bar\lambda\, z^{\mu *} = \bar\lambda\, \Lambda^\mu_\nu\, z^\nu = \bar\lambda^2\, \Lambda^\mu_\nu\, \tilde z^\nu$. This shows that φ is a homomorphism

$$\varphi : C_{\rm alg}(S^3(\Lambda')) \to C_{\rm alg}(S^3(\Lambda))$$

and that $\varphi([S^3(\Lambda')]) = [S^3(\Lambda)]$.

2) One has $\mathrm{Spin}(4) = SU(2) \times SU(2)$ and the covering map

$$\pi\ :\ \mathrm{Spin}(4) = SU(2) \times SU(2) \to SO(4)$$

is given for any $(u,v) \in SU(2) \times SU(2)$ and $\xi \in \mathbb{R}^4$ by $\pi(u,v)\xi = \eta$ with

$$\tau_\mu\, \eta^\mu =\ u\,(\tau_\mu\, \xi^\mu)\, v^*\,. \tag{2.20}$$

This equality continues to hold for the natural complex linear extension $V^\mu_\nu\, z^\nu$ of $V = \pi(u,v)$ to $\mathbb{C}^4$ and it follows that with the notations of assertion 2) one has

$$\tau_\mu\, V^\mu_\nu\, z^\nu_\Lambda =\ u\, U\, v^*$$

with $U = U_\Lambda$ as above. The unitary $u\,U\,v^*$ still fulfills $\mathrm{ch}_{1/2}(u\,U\,v^*) = 0$ and $\mathrm{ch}_{3/2}(u\,U\,v^*) = \mathrm{ch}_{3/2}(U)$ since in $M_2(\mathbb{C})$ it is the ordinary product and trace which are involved in the formulas for $\mathrm{ch}_{k/2}$. Thus, with $\tilde z^\mu = V^\mu_\nu\, z^\nu$ we just have to check that $\tilde z^{\mu *} = \Lambda'^{\mu}_\nu\, \tilde z^\nu$. One has $\Lambda'\,\tilde z = V\Lambda V^t\, V z = V\Lambda\, z = V z^*$ and since V has *real* coefficients this is $(Vz)^* = (\tilde z)^*$.

3) Let $U = U_\Lambda$ as above, then U^* is still unitary and fulfills $\mathrm{ch}_{1/2}(U^*) = 0$, $\mathrm{ch}_{3/2}(U^*) = -\,\mathrm{ch}_{3/2}(U)$. It corresponds to $\tilde z^0 = z^{0*}$, $\tilde z^k = -z^{k*}$. Then $\tilde z^{\mu *} = \varepsilon_\mu\, z^\mu = \varepsilon_\mu(\Lambda^{-1})^\mu_\nu\, z^{\nu *} = \varepsilon_\mu(\Lambda^{-1})^\mu_\nu\, \varepsilon_\nu\, \tilde z^\nu$ which gives the value of Λ'. ☐

Corollary 2.5. *For every $\Lambda \in \mathcal{S}$ there exists $\varphi_j \in \mathbb{R}/\pi\mathbb{Z}$ and isomorphisms $S^3(\Lambda) \to S^3_\varphi$ (resp. $\mathbb{R}^4(\Lambda) \to \mathbb{R}^4_\varphi$) where S^3_φ corresponds to the diagonal matrix*

$$\Lambda(\varphi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-2i\varphi_k} \end{bmatrix}$$

Proof. We just need to recall why the matrix Λ can be diagonalized by a $W \in SO(4)$. In fact Λ is unitary and fulfills $\Lambda^t = \Lambda$ *i.e.* $\Lambda^* = J\Lambda\, J^{-1}$ where J gives the real structure of $\mathbb{C}^4$. An eigenspace $E_\lambda = \{\xi \in \mathbb{C}^4; \Lambda\,\xi = \lambda\,\xi\}$ is stable by J since $\Lambda\, J^{-1}\,\xi = J^{-1}\,\Lambda^*\,\xi = J^{-1}\,\bar\lambda\,\xi = \lambda\, J^{-1}\,\xi$ (and $J^{-1} = J$). Thus we can find an orthonormal basis of *real* vectors: $J\,\xi_i = \xi_i$ which are eigenvectors for Λ. With e_i the standard basis of $\mathbb{C}^4$ the map $e_i \to \xi_i$ gives an element of $O(4)$ and we can take it in $SO(4)$. Thus $\Lambda = W\,D\,W^t$ with $W \in SO(4)$. ☐

The presentation of the algebra of $\mathbb{R}^4_\varphi$ is given by (2.14) (2.15) and the relation (2.7) with $\Lambda = \Lambda(\varphi)$. This gives $z^{0*} = z^0$ and $z^{k*} = e^{-2i\varphi_k}\, z^k$ so that with $x^0 = z^0$, $x^k = e^{-i\varphi_k}\, z^k$ we get

$$x^{\mu *} = x^\mu\,, \quad \forall \mu \in \{0,1,2,3\}\,. \tag{2.21}$$

and the six other relations give

$$e^{i\varphi_k}\, x^k\, x^0 - e^{-i\varphi_k}\, x^0\, x^k + \sum \varepsilon_{k\ell m}\, e^{i(\varphi_\ell - \varphi_m)}\, x^\ell\, x^m = 0 \tag{2.22}$$

$$e^{i\varphi_k}\, x^0\, x^k - e^{-i\varphi_k}\, x_k\, x^0 + \sum \varepsilon_{k\ell m}\, e^{-i(\varphi_\ell - \varphi_m)}\, x^\ell\, x^m = 0\,. \tag{2.23}$$

This gives, by combining (2.22) and (2.23) the relations

$$\sin(\varphi_k)\,[x^0, x^k]_+ = i\, \cos(\varphi_\ell - \varphi_m)\,[x^\ell, x^m] \tag{2.24}$$

$$\cos(\varphi_k)\,[x^0, x^k] = i\, \sin(\varphi_\ell - \varphi_m)\,[x^\ell, x^m]_+\,, \tag{2.25}$$

where we let $[a,b]_+ = \, a\,b + \, b\,a$ be the anticommutator.

We shall now study in much greater detail the corresponding moduli space.

3. The real moduli space $\mathcal{M}$

We let $\mathcal{M}$ be the moduli space of noncommutative 3-spheres. It is obtained from proposition 2.4 1) and 2) as the quotient

$$\mathcal{M} = \, (U(1) \times SO(4)) \backslash \mathcal{S}\,, \tag{3.1}$$

of $\mathcal{S}$ by the action of $U(1) \times SO(4)$. This action of $U(1) \times SO(4)$ on $\mathcal{S}$ is the restriction of the following action of $U(4)$ on $\mathcal{S}$.

$$\Lambda \in \mathcal{S} \to W\, \Lambda\, W^t\,, \quad \forall\, W \in U(4) \tag{3.2}$$

which allows to identify $\mathcal{S}$ with the homogeneous space

$$U(4)/O(4) \simeq \mathcal{S}\,. \tag{3.3}$$

(Note that any $\Lambda \in \mathcal{S}$ can be written as $\Lambda = V V^t$ for some $V \in U(4)$ since it can be diagonalized by an orthogonal matrix). The presence of $U(1)$ in (3.1) allows to reduce to $SU(4)$ and one obtains this way a first convenient description of $\mathcal{M}$.

3.1. $\mathcal{M}$ in terms of A_3.

The description of $\mathcal{M}$ in terms of the compact Lie group $SU(4)$ is given by :

Proposition 3.1. 1) *Let N be the normalizer of $SO(4)$ in $SU(4)$. Then one has a canonical isomorphism*

$$\mathcal{M} \simeq SO(4)\backslash SU(4)/N\,. \tag{3.4}$$

2) *Let $\mathbb{T} \subset SU(4)$ be the maximal torus of diagonal matrices, and $W \subset \mathrm{Aut}(\mathbb{T})$ the corresponding Weyl group. Let $D = \mathbb{T} \cap N$. Then the above restricts to an isomorphism*

$$\mathcal{M} \simeq W\backslash \mathbb{T}/D\,. \tag{3.5}$$

3) *The map $u \to u^2$ from $\mathbb{T}$ to $\mathbb{T}$ induces an isomorphism*

$$\mathcal{M} \simeq W\backslash \mathbb{T}/D \simeq \text{ Space of Conjugacy Classes in } PSU(4)\,. \tag{3.6}$$

Proof. 1) Let Z be the center of $SU(4)$, it is generated by i which has order 4 and contains $-1 \in SO(4)$. The normalizer N is generated by $SO(4)$ and the element

$$v = w \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}, \qquad w = e^{2\pi i/8}\,, \tag{3.7}$$

which implements the outer automorphism of $SO(4)$ and whose square $v^2 = i$ generates Z.
Let $X = U(1)\backslash \mathcal{S}$ considered as an homogeneous space on $SU(4)$ using the action (3.2) *i.e.* $\Lambda \in X \to W\,\Lambda\,W^t$. Given $\Lambda \in \mathcal{S}$ we can find $\lambda \in U(1)$ so that $\Lambda = \lambda\,\Lambda_1$ with $\mathrm{Det}\,\Lambda_1 = 1$, thus with $\mathcal{S}_1 = \{\Lambda \in \mathcal{S}, \mathrm{Det}\,\Lambda = 1\}$ one has,

$$X = U(1)\backslash \mathcal{S} \simeq w^{\mathbb{Z}}\backslash \mathcal{S}_1 \simeq SU(4)/N\,. \tag{3.8}$$

Indeed the first equality follows since the action (3.2) of w is multiplication by i and the second follows by computing the isotropy group K of $1 \in w^{\mathbb{Z}}\backslash \mathcal{S}_1$. One has $SO(4) \subset K$ and $v \in K$ since $v^t v = v^2 = w^2$. Thus $N \subset K$. Conversely given $V \in K$ one has $V V^t = i^N$ for some $N \in \mathbb{Z}$ and thus for a suitable power of v one has $v^k\,V\,V^t\,v^k = 1$ thus $v^k\,V \in SO(4)$. This shows that $K = N$ and we get the first statement of the proposition since $\mathcal{M} \simeq SO(4)\backslash X$ by construction.
Note the standard description of N which is given as follows. One lets $\theta \in \mathrm{Aut}(SU(4))$ be given by complex conjugation,

$$\theta(u) = \bar{u} = J\,u\,J^{-1} \qquad \forall\, u \in SU(4)\,. \tag{3.9}$$

One has $\theta^2 = 1$ and the fixed points of θ give $SO(4) = SU(4)^\theta$. The normalizer N of $SO(4)$ is characterized by,

$$u^{-1}\,\theta(u) \in Z\,, \tag{3.10}$$

where Z is the center of $SU(4)$.

2) Let σ be the map $\sigma(u) = u\,u^t$ from $SU(4)$ to the space $P\mathcal{S}_1$ of classes of elements of $\mathcal{S}_1$ modulo the action of Z by multiplication. It follows from 1) that σ is an isomorphism of $X = SU(4)/N$ with $P\mathcal{S}_1$. Given $u \in SU(4)$ we can find $V \in SO(4)$ such that

$$u\,u^t = V D\,V^t \tag{3.11}$$

where D is a diagonal matrix. Then $\sigma(u) = \sigma(VD^{1/2})$ where $D^{1/2}$ is a diagonal square root of D with determinant equal to 1. Thus every element of the coset space $SO(4)\backslash X$ can be represented by a diagonal matrix, and the natural map given by inclusion

$$W\backslash\mathbb{T}/D \to SO(4)\backslash X \tag{3.12}$$

is surjective.

3) When restricted to $\mathbb{T}$ the map σ is simply the squaring $u \to u^2$. Moreover the equality $\sigma(u_1) = \sigma(u_2)$ in $P\mathcal{S}_1$ for $u_j \in \mathbb{T}$ just means that the u_j^2 define the same element of the maximal torus $\mathbb{T}/Z$ of $PSU(4)$. Thus the result follows. Let us check that the group D/Z is $(\mathbb{Z}/2)^3$. The elements of D are the $v^k\, u$ with v as in (3.7) and u in $\mathbb{T} \cap SO(4)$. One has $v^2 \in Z$ and modulo $Z \cap SO(4) = \pm 1$ the elements of $\mathbb{T} \cap SO(4)$ form the Klein group $H = (\mathbb{Z}/2\mathbb{Z})^2$. Thus one gets $D/Z \simeq (\mathbb{Z}/2\mathbb{Z})^3$. □

We identify the Lie algebra of $SU(4)$ with the Lie algebra of antihermitian matrices with trace 0,

$$\mathrm{Lie}\,(SU(4)) = \{T \in M_4(\mathbb{C})\,;\; T^* = -T\,,\ \mathrm{Trace}\,T = 0\} \tag{3.13}$$

where θ is still acting by complex conjugation.
The diagonal matrices $D \in \mathcal{D}$ form a maximal abelian Lie subalgebra of the eigenspace,

$$\mathrm{Lie}\,(SU(4))^- = \{T\,,\ \theta(T) = -T\}\,. \tag{3.14}$$

The roots $\alpha \in \Delta$ are given by,

$$\alpha_{\mu,\nu}(\delta) = \delta_\mu - \delta_\nu\,. \tag{3.15}$$

Proposition 3.2. *For $\delta \in \mathcal{D}$ one has,*

$$e^\delta \in N \;\Leftrightarrow\; e^{2\delta} \in Z \;\Leftrightarrow\; \alpha_{\mu,\nu}(\delta) \in i\,\pi\,\mathbb{Z}\,, \quad \forall \mu, \nu\,. \tag{3.16}$$

Proof. The equivalence between the last two conditions is a general fact for compact Lie groups (*cf.* [7]). The equivalence between the first two conditions follows from the third statement of proposition 3.1. □

We let Γ be the lattice $\Gamma \subset \mathcal{D}$ determined by the equivalent conditions (3.16), and

$$\mathbb{T}_A = \mathcal{D}/\Gamma \tag{3.17}$$

be the quotient 3-dimensional torus.

We let the group W of permutations of 4 elements act on $\mathbb{T}_A$ by permutations of the δ_μ. In fact we view it as the Weyl group of the pair $(SU(4), N)$, *i.e.* as the quotient,

$$W = \mathcal{N}/C \tag{3.18}$$

of the normalizer of $\mathcal{D}$ in $SO(4)$ by the centralizer of $\mathcal{D}$. Note that v being diagonal is in the centralizer of $\mathcal{D}$ so that W does not change in replacing $SO(4)$ by N since $v^k\,u$ normalizes $\mathcal{D}$ iff u does.

Corollary 3.3. *The map $\sigma(\delta) = e^{2\delta}$ defines an isomorphism of the quotient of $\mathbb{T}_A$ by the action of W with the moduli space $\mathcal{M} = (U(1) \times SO(4))\backslash\mathcal{S}$.*

Proof. This is just another way to write the second statement of proposition 3.1. □

3.2. **Trigonometric parameters φ of S^3_φ.**

We shall now describe a convenient parametrization of the torus $\mathbb{T}_A$ which gives the corresponding algebras in the form of corollary 2.5. It is given by the map

$$\varphi = (\varphi_1, \varphi_2, \varphi_3) \in (\mathbb{R}/\pi\,\mathbb{Z})^3 \to d(\varphi) = (\alpha_0, \alpha_0 - i\,\varphi_1, \alpha_0 - i\,\varphi_2, \alpha_0 - i\,\varphi_3)\,, \quad \alpha_0 = \frac{i}{4}\sum \varphi_j\,. \tag{3.19}$$

One has $\alpha_{0,k}(d(\varphi)) = i\,\varphi_k$ and the definition of Γ shows that d is an isomorphism. Also

$$e^{2d(\varphi)} \simeq \begin{bmatrix} 1 & 0 \\ 0 & e^{-2i\varphi_k} \end{bmatrix}$$

up to multiplication by a scalar.

In terms of the parameters φ_j the twelve roots $\alpha \in \Delta$ are the following linear forms,

$$(\varphi_1, \varphi_2, \varphi_3) \to i\,\{\pm\varphi_j, \varphi_k - \varphi_l\}\,. \tag{3.20}$$

The action of the Weyl group W gives the following linear transformations of the φ_j. Arbitrary permutations of the φ_j's correspond to permutations of the last three α_k's. The transposition of α_0 with α_1 corresponds to :

$$T_{01}(\varphi_1, \varphi_2, \varphi_3) = (-\varphi_1, \varphi_2 - \varphi_1, \varphi_3 - \varphi_1)\,. \tag{3.21}$$

The 3-spheres S^3_φ are parametrized by

$$\varphi = (\varphi_1, \varphi_2, \varphi_3) \in (\mathbb{R}/\pi\mathbb{Z})^3 \tag{3.22}$$

modulo the action of the Weyl group $W \simeq S_4$ generated by the permutation group S_3 of the φ_j's and

$$(\varphi_j) \to (-\,\varphi_1\,,\ \varphi_3 - \varphi_1\,,\ \varphi_2 - \varphi_1) = (\varphi'_j)\,. \tag{3.23}$$

3.3. **$\mathcal{M}$ in terms of D_3.**

To obtain another very convenient parametrization of the torus $\mathbb{T}_A$ we use the isomorphism $A_3 \sim D_3$, *i.e.*

$$SU(4) \simeq \mathrm{Spin}\,(6)\,. \tag{3.24}$$

which simply comes from the spin representation of Spin (6). The Clifford algebra $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^6)$ has dimension 2^6 and is a matrix algebra $M_n(\mathbb{C})$ with $n = 2^3 = 8$. We let γ^μ be the corresponding γ-matrices, with

$$\gamma_\mu^* = \gamma_\mu\,, \quad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = \,2\,\delta_{\mu,\nu}$$

We then let

$$\sigma_{\mu\nu} = \frac{1}{2}\,(\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu)$$

which span a real subspace of the Clifford algebra stable under bracket and isomorphic (up to a factor of 2) to the Lie algebra of $SO(6)$ of real antisymmetric 6 by 6 matrices.

Proposition 3.4. 1) *The following gives a parametrization τ of a maximal torus in* Spin(6)*,*

$$\theta = (\theta_j) \in (\mathbb{R}/\,2\pi\mathbb{Z})^3 \xrightarrow{\tau} \mathrm{Exp}\sum \theta_j\sigma_{2j-1,2j} \in \mathrm{Spin}(6) \subset \mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^6) \tag{3.25}$$

2) *The half Spin representation gives an isomorphism $\pi\ :\ \mathrm{Spin}(6) \to SU(4)$.*

3) *One has* $\pi(\tau(\theta)) = e^{\delta_\theta}$ *with* δ_θ *diagonal given by*

$$\delta_\theta = i\,(\theta_1 + \theta_2 + \theta_3\,,\; \theta_1 - \theta_2 - \theta_3\,,\; -\theta_1 + \theta_2 - \theta_3\,,\; -\theta_1 - \theta_2 + \theta_3)\,. \tag{3.26}$$

Proof. This is a straightforward check, since in the half Spin representation π one has in a suitable basis

$$\sigma_{12} \to \begin{bmatrix} i & & & 0 \\ & i & & \\ & & -i & \\ 0 & & & -i \end{bmatrix}, \quad \sigma_{34} \to \begin{bmatrix} i & & & 0 \\ & -i & & \\ & & i & \\ 0 & & & -i \end{bmatrix}, \quad \sigma_{56} \to \begin{bmatrix} i & & & 0 \\ & -i & & \\ & & -i & \\ 0 & & & i \end{bmatrix}, \tag{3.27}$$

thus $\pi(\tau(\theta)) = e^{\delta_\theta}$ with δ_θ given by (3.26). □

Thus the trigonometric parameters φ_k are given in terms of the θ's by,

$$\varphi_1 = 2\,(\theta_2 + \theta_3)\,,\; \varphi_2 = 2\,(\theta_1 + \theta_3)\,,\; \varphi_3 = 2\,(\theta_1 + \theta_2)\,. \tag{3.28}$$

The natural parameters for the maximal torus $\mathbb{T}_D$ of $SO(6)$ are the $\psi_j = 2\,\theta_j$ and are defined modulo $2\,\pi\,\mathbb{Z}$, *i.e.* correspond to the Lie algebra element

$$\ell(\psi) = \psi_1\,\beta_{12} + \psi_2\,\beta_{34} + \psi_3\,\beta_{56} \tag{3.29}$$

where the β_{ij} form the canonical basis of real antisymmetric matrices. The kernel of the covering $\mathrm{Spin}\,(6) \to SO(6)$ corresponds to $\theta_j = \pi$ so that for $SO(6)$ the torus $\mathbb{T}_D$ is parametrized by the ψ_j's defined modulo 2π by

$$\psi \in (\mathbb{R}/2\,\pi\,\mathbb{Z})^3 \to e^{\ell(\psi)}\,. \tag{3.30}$$

The transition from ψ to φ's is given then by,

$$\varphi_j = \psi_k + \psi_\ell\,,\; 2\,\psi_j = \varphi_k + \varphi_\ell - \varphi_j\,. \tag{3.31}$$

as well as

$$\varphi_1 - \varphi_2 = \psi_2 - \psi_1\,,\; \varphi_2 - \varphi_3 = \psi_3 - \psi_2\,,\; \varphi_3 - \varphi_1 = \psi_1 - \psi_3\,. \tag{3.32}$$

We shall now spell in great details the basic Lie group datas for D_3 and get a description of the moduli space $\mathcal{M}$ in these terms. Proposition 3.1 3) gives a natural isomorphism

$$\mathcal{M} \simeq \text{ Space of Conjugacy Classes in } PSU(4)$$

of the moduli space $\mathcal{M}$ with the space of conjugacy classes of elements of $PSU(4) \simeq PSO(6)$. The general theory of compact Lie groups provides a natural triangulation of such a space of conjugacy classes in terms of alcoves. The latter are obtained as the connected components of the complement of the union of the singular hyperplanes.

Our aim in this section is to describe such a triangulation in our specific case and to exhibit the role of the singular hyperplanes. We shall see in the next section their natural compatibility with the scaling foliation.

In terms of the parameters φ_j of the 3-spheres S^3_φ the relations which specify the non generic situations, are all of the form

$$\Big\{\varphi, \alpha(\varphi) = n\,\frac{\pi}{2}\Big\} = G_{\alpha,n} \tag{3.33}$$

where n is an integer and α is one of the twelve "roots" *i.e.* $\alpha \in \Delta = \pm\{\varphi_1, \varphi_2, \varphi_3, \varphi_1 - \varphi_2, \varphi_2 - \varphi_3, \varphi_3 - \varphi_1\}$.

Moreover the periodicity lattice of φ is $(\pi\mathbb{Z})^3$ which is specified by

$$\{\varphi, \alpha(\varphi) \in \pi\mathbb{Z}\,,\ \forall\, \alpha \in \Delta\} = \Gamma_\varphi\,. \tag{3.34}$$

We now want to relate more precisely the above situation with canonical objects (root systems, alcoves, chambers, affine Weyl group, nodal vectors ...) associated to the following data $(G, \mathbf{T})$

$$G = PSO(6)\,,\ \mathbf{T} = \text{Maximal torus } \mathbb{T}_D/\pm 1\ . \tag{3.35}$$

We use the natural parametrization of the Lie algebra Lie $(\mathbf{T})$,

$$\ell(\xi) = \ \xi_1\, \beta_{12} + \xi_2\, \beta_{34} + \xi_3\, \beta_{56}\,. \tag{3.36}$$

Since we used the "squaring" $u \to u^2$ in the isomorphism of proposition 3.1 3), the parameter ψ that appears in the transition $\varphi \to \psi$ of equation (3.31) is related to ξ by

$$\xi = \ 2\,\psi\,. \tag{3.37}$$

In other words the natural relation between the parameters φ and ξ is

$$2\,\varphi_j = \xi_k + \xi_\ell\,,\ \xi_j = \varphi_k + \varphi_\ell - \varphi_j\,. \tag{3.38}$$

This accounts for a factor of $\frac{1}{2}$ in (3.33) but does not yet relate it to the equation of singular hyperplanes. To understand this relation more precisely we shall now review briefly the standard ingredients of the theory of alcoves for the specific data $(G, \mathbf{T})$.

3.4. Roots $\Delta = R(G, \mathbf{T})$.

They are by definition the linear forms α on Lie $(\mathbf{T})$ given by eigenvalues $X_\alpha \in \mathrm{Lie}\, G_{\mathbb{C}}$ which fulfill:

$$[\xi, X_\alpha] = \alpha(\xi)\, X_\alpha \qquad \forall\, \xi \in \mathrm{Lie}\, \mathbf{T} \subset \mathrm{Lie}\, G\,. \tag{3.39}$$

They are complex valued as defined. In our case they are given by (up to multiplication by i)

$$(\pm\, e_\mu\ \pm'\, e_\nu)\, \xi = \pm\, \xi_\mu\ \pm'\, \xi_\nu\,. \tag{3.40}$$

3.5. Singular hyperplanes $H_{\alpha,n}$.

They are given by a root $\alpha \in \Delta$ and $n \in \mathbb{Z}$, with

$$H_{\alpha,n} = \{\xi, \alpha(\xi) = 2\pi i n\}\,. \tag{3.41}$$

As we shall explain below while these hyperplanes suffice to obtain a triangulation of the space of conjugacy classes of the simply connected covering of G we shall need the additional ones with $n \in \frac{1}{2}\mathbb{Z}$ to describe the space of conjugacy classes in G itself.

3.6. Kernel of the exponential map: $\Gamma(\mathbf{T})$ (nodal group of $\mathbf{T}$).

In our case we are dealing with $G = PSO(6)$ and thus for $\xi \in \mathrm{Lie}\, \mathbf{T}$, e^ξ is 1 in G iff $Ad(e^\xi) = 1$ since the center of G is $C(G) = \{1\}$. This means exactly that all eigenvalues of $ad(\xi)$ belong to Ker (exp) $= 2\pi i\mathbb{Z}$, thus

$$\Gamma(\mathbf{T}) = \{\xi, \alpha(\xi) \in 2\pi i\mathbb{Z} \quad \forall\, \alpha \in \Delta\}\,. \tag{3.42}$$

Thus, after applying the change of variables (3.38)

$$\Gamma_\varphi \simeq \Gamma(\mathbf{T}) \tag{3.43}$$

which shows that the periodicity lattice we have is the nodal group of $\mathbf{T}$.

3.7. **Group of nodal vectors** $N(G,\mathbf{T})\subset\Gamma(\mathbf{T})$.

This group can be defined in terms of vectors K_α which are associated to the roots $\alpha\in\Delta$ but in our case it is simpler to use the definition as follows:

$$N(G,\mathbf{T}) = \text{Kernel of } \exp : \text{Lie}\,\mathbf{T}\to\tilde{G} = \text{Universal cover of } G\,. \tag{3.44}$$

In our case $\tilde{G}=\text{Spin}\,6$ and in terms of γ-matrices the exponential map takes the form,

$$\begin{aligned}\ell(\xi)\to\qquad & \left(\cos\frac{1}{2}\,\xi_1+\sin\frac{1}{2}\,\xi_1\,\gamma_1\gamma_2\right)\left(\cos\frac{1}{2}\,\xi_2+\sin\frac{1}{2}\,\xi_2\,\gamma_3\gamma_4\right)\\ & \left(\cos\frac{1}{2}\,\xi_3+\sin\frac{1}{2}\,\xi_3\,\gamma_5\gamma_6\right)\end{aligned} \tag{3.45}$$

whose kernel is given by

$$N(G,\mathbf{T}) = \{\xi\,;\ \xi_j = 2\pi n_j\,,\quad n_j\in\mathbb{Z}\,,\quad n_j\ \textit{even}\}\,. \tag{3.46}$$

One has $N(G,\mathbf{T})\subset\Gamma(\mathbf{T})$ and the quotient, of order 4, is generated by $\xi=(\pi,\pi,\pi)\in\Gamma(\mathbf{T})$.

3.8. **Affine Weyl group** W_a.

The affine Weyl group W_a is the group generated by the reflexions associated to the hyperplane $H_{\alpha,n}$ for $\alpha\in\Delta$ and $n\in\mathbb{Z}$.

One has [6] (Chapter VI, proposition 1, page 173)

$$W_a = N(G,\mathbf{T})\rtimes W \tag{3.47}$$

where the Weyl group is generated by the reflexions associated to singular hyperplanes $H_{\alpha,0}$.

In our case $W=S_4$ and all its elements are of the form

$$W=\varepsilon\,\sigma\,,\quad \sigma\in S_3\,,\quad \varepsilon=(\varepsilon_i)\,,\quad \varepsilon_i\in\pm1\,,\quad \prod\varepsilon_i = 1 \tag{3.48}$$

where the action on ξ is by permutation of the ξ_j for σ (careful that $(\sigma\xi)(i)=\xi(\sigma^{-1}(i))$ to get a covariant action) and by multiplication by ε_i for ε.

For $N(G,\mathbf{T})$ we can check that the $\check{\alpha}=K_\alpha$ are simply given by the vectors $\pm\,e_\mu\pm' e_\nu$ which correspond to

$$(\pm\,2\pi,\pm'\,2\pi,0) = K_{\pm e_1\pm' e_2}\,,\quad \langle K_\alpha,\alpha\rangle = 2\,. \tag{3.49}$$

3.9. **Affine Weyl group** W'_a.

It is by definition the semi direct product,

$$W'_a = \Gamma(\mathbf{T})\rtimes W\,. \tag{3.50}$$

What matters is that it still acts on the set of singular hyperplanes. For $\gamma\in\Gamma(\mathbf{T})$ one has $\alpha(h+\gamma)=\alpha(h)+\alpha(\gamma)$ and $\alpha(\gamma)\in 2\pi i\mathbb{Z}$ thus one is just shifting the n in $H_{\alpha,n}$.

From [7] proposition 2 (Chapter 9, page 45) W_a is a normal subgroup of W'_a.

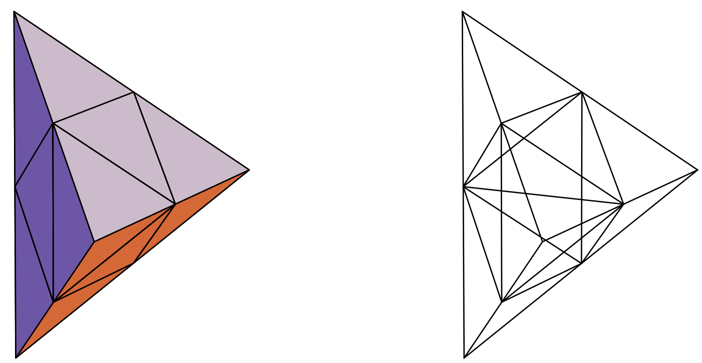

FIGURE 1. Tiling of the alcove X by $\frac{1}{2}$-alcoves

3.10. **Alcoves and fundamental domain.**

The alcoves are the connected components of $(UH_{\alpha,n})^c \subset \mathrm{Lie}\,\mathbf{T}$. The chambers are the components of $(UH_{\alpha,0})^c$.

By construction the alcoves are intersections of half spaces and are thus convex polyhedra.

By [7] the affine Weyl group W_a acts simply transitively on $\Sigma =$ the set of alcoves. Since W'_a is still acting on Σ we can identify Σ with the homogeneous space

$$\Sigma = W'_a/H_X \tag{3.51}$$

where H_X is the finite isotropy group of an alcove X.

In our case, we take the following alcove :

$$X = \{\xi, \xi_1 + \xi_2 \geqq 0, \xi_2 - \xi_1 \geqq 0, \xi_3 - \xi_2 \geqq 0, \xi_2 + \xi_3 \leqq 2\pi\}\,. \tag{3.52}$$

It is a tetrahedron with all 4-faces congruent but 2 long edges and 4 short edges.

Lemma 3.5. *The isotropy subgroup $H_X \subset W'_a$ of X is generated by $w_1 = \left(\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right),\, \varepsilon_{12}\,\sigma_{13}\right)$.*

Proof. For convenience we rescale the ξ_j by 2π. The coordinates of the vertices of X are then 0, $p = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $q = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $p' = (0,0,1)$.

One has $w_1(0) = p$, $w_1(p) = p + \varepsilon_{12}\,\sigma_{13}(p) = p + \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) = p'$, $w_1(p') = p + \varepsilon_{12}\,\sigma_{13}(p') = p + (-1,0,0) = q$, $w_1(q) = p + \varepsilon_{12}\,\sigma_{13}(q) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = 0$. □

We let σ be the orthogonal reflexion around the face (pqp') of X. One has $\sigma \in W_a$ and it is given explicitly by

$$\sigma = ((0,1,1), \varepsilon_{23}\,\sigma_{23})\,. \tag{3.53}$$

Indeed, since the face is given by $\xi_2 + \xi_3 = 2\pi$ the corresponding nodal vector is $(0,1,1)$. One checks that $\sigma^2 = 1$ and that σ fixes p, q, p', $(\sigma(p') = (0,1,1) + (0,-1,0) = (0,0,1) = p')$.

We let $Y = \sigma(X)$ be the reflexion of X along that face which yields the convex pentahedra $X \cup Y$.

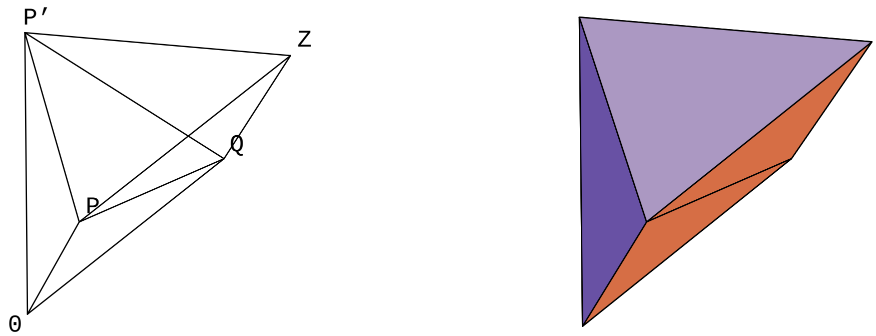

FIGURE 2. Fundamental domain

Proposition 3.6. *Let X be the alcove given by* (3.52) *and $Y = \sigma(X)$ its reflexion along the face (p, q, p'). Then $\frac{1}{2}(X \cup Y)$ is a fundamental domain for the action of $\Gamma(\mathbf{T}) \rtimes W$ in* $\mathrm{Lie}\,\mathbf{T}$.

Proof. By the above lemma the action of W'_a on the set of alcoves is the same as the action of W'_a on W'_a/H_X. Let $W''_a = 2\Gamma \rtimes W$. The left coset space W''_a/W'_a is identified with the 8 elements set, $\Gamma/2\Gamma$ where the corresponding map is given by:

$$(\gamma, w) \in W'_a \to \text{Class of } w^{-1}(\gamma) \text{ in } \Gamma/2\Gamma\,. \tag{3.54}$$

Indeed $(0, w^{-1})(\gamma, w) = (w^{-1}(\gamma), 1)$. We can thus display the action (on the right) of H_X as $\gamma \to$ Class of $\sigma_{13}\,\varepsilon_{12}(p + \gamma)$. Let us write this transformation in terms of the φ-coordinates. One obtains

$$\varphi \to w\left(\varphi + \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right)\,, \quad w(\varphi) = (-\varphi_3, \varphi_1 - \varphi_3, \varphi_2 - \varphi_3)\,. \tag{3.55}$$

Thus in fact we look at the following transformation of $(\mathbb{Z}/2)^3$,

$$S(a_1, a_2, a_3) = \left(\frac{1}{2} - a_3, a_1 - a_3, a_2 - a_3\right)\,. \tag{3.56}$$

One has $S(0) = \left(\frac{1}{2}, 0, 0\right)$, $S\left(\frac{1}{2}, 0, 0\right) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$, $S\left(\frac{1}{2}, \frac{1}{2}, 0\right) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $S\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 0$. The other orbit is $S\left(0, \frac{1}{2}, 0\right) = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$, $S\left(\frac{1}{2}, 0, \frac{1}{2}\right) = \left(0, 0, \frac{1}{2}\right)$, $S(0, 0, \frac{1}{2}) = \left(0, \frac{1}{2}, \frac{1}{2}\right)$, $S\left(0, \frac{1}{2}, \frac{1}{2}\right) = \left(0, \frac{1}{2}, 0\right)$. This shows that the double coset space $W''_a \backslash W'_a/H_X$ has cardinality 2 and thus that W''_a just has 2 orbits in its action on the set of alcoves. What remains is to show that $Y \notin$ Orbit of X. But one has $Y = \sigma(X)$ with σ given by (3.53). Thus we just need to determine the double coset of σ, *i.e.* by (3.54) the class of $\sigma_{23}\,\varepsilon_{23}(0, 1, 1)$ in $\Gamma/2\Gamma$. One just checks that it is in the other orbit. □

We thus have a fairly simple fundamental domain for the parameter space of noncommutative 3-spheres. In terms of the φ's a simple translation of the above result gives,

Proposition 3.7. *1) The union $A \cup B$ of the following simplices*

$$A = \left\{\varphi\,;\ \frac{\pi}{2} \geqq \varphi_1 \geqq \varphi_2 \geqq \varphi_3 \geqq 0\right\}\,, \quad B = \left\{\varphi\,;\ \varphi_3 + \frac{\pi}{2} \geqq \varphi_1 \geqq \frac{\pi}{2} \geqq \varphi_2 \geqq \varphi_3\right\}\,, \tag{3.57}$$

gives a fundamental domain for the action of W on $\mathbb{R}^3/\Gamma_\varphi = (\mathbb{R}/\pi\mathbb{Z})^3$.

2) The real moduli space $\mathcal{M}$ is obtained by glueing the face[3] $(P'QZ)$ *to the face* (ZPP') *by the transformation* $\gamma \in \Gamma_\varphi \rtimes W$

$$\gamma(\varphi_1, \varphi_2, \varphi_3) = (\pi - \varphi_1 + \varphi_2,\, \pi - \varphi_1 + \varphi_3,\, \pi - \varphi_1)$$

and crossing each wall, i.e. each face whose supporting hyperplane contains 0*, by the corresponding reflexion (in* W*).*

The two simplices A and B have a common face (PQP') supported by the hyperplane $\varphi_1 = \frac{\pi}{2}$. In order to describe the moduli space one needs to give the identifcations of the boundary components. All faces of A other than (PQP') are walls of chambers and thus crossing them leads to a simple reflexion on that wall.

4. The flow F

While the space $\mathcal{M}$ is the natural moduli space for noncommuative 3-spheres, the moduli space of the corresponding 4-spaces $\mathbb{R}^4_\varphi$ is obtained as a space of flow lines for a natural flow F on $\mathcal{M}$. We define this flow F as the gradient flow for the Killing metric on the Lie algebra of $SO(6)$ of the character of the virtual representation given by the "signature". We first show that F is nicely compatible with the triangulation by alcoves and then check that the isomorphism class of the 4-spaces $\mathbb{R}^4_\varphi$ is constant along the flow lines. We shall need for the converse to have a complete knowledge of the geometric datas of these quadratic algebras, and this will be obtained in section 5. Thus the converse will be proved later on in section 6.

4.1. Compatibility of F with the triangulation by alcoves.

The basic compatibility of the flow F with the structure of the real moduli space $\mathcal{M}$ is given by :

Proposition 4.1. a) *The character of the signature representation is*

$$\chi(\xi) = \mathrm{Trace}\,(*\,\pi(\ell(\xi)) = -\,8 \prod \sin \xi_j\,. \tag{4.1}$$

b) *The flow* $F = \nabla\, \chi(2\psi)$ *is invariant under the action of the Weyl group* W *and leaves each of the singular hyperplanes* $H_{\alpha,n}$*,* $n \in \frac{1}{2}\mathbb{Z}$ *globally invariant.*

c) *In terms of the variables* φ_j *one has*

$$F = \sum \sin(2\varphi_j) \sin(\varphi_k + \varphi_\ell - \varphi_j)\, \frac{\partial}{\partial \varphi_j}\,. \tag{4.2}$$

Proof. a) We parametrize the maximal torus $\mathbf{T}_D$ of $SO(6)$ by the ξ_j as in (3.29) *i.e.*

$$\ell(\xi) = \xi_1\, \beta_{12} + \xi_2\, \beta_{34} + \xi_3\, \beta_{56}\,.$$

The Killing metric is simply given there (up to scale and sign) by $\sum d\xi_j^2$. We identify $\wedge^3 \mathbb{C}^6$ with the linear span of the $e_{ijk} = \gamma_i\, \gamma_j\, \gamma_k$, $\mathrm{card}\,\{i,j,k\} = 3$ in $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^6)$. We then use $\gamma_7 = \gamma_1\, \gamma_2\, \gamma_3\, \gamma_4\, \gamma_5\, \gamma_6$, which fulfills $\gamma_7^2 = -1$ to define the $*$ operation by:

$$*\, e_{ijk} = \gamma_7\, e_{ijk}\,. \tag{4.3}$$

One checks that the matrix of $*\,\pi(\ell(\xi))$ restricted to the 12 dim subspace spanned by the e_{ijk} where two indices belong to one of the 3 subsets $\{1,2\}$, $\{3,4\}$, $\{5,6\}$ is off diagonal and hence has vanishing

[3] we use capital letters P, Q, Q' for the vertices : $P = (\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$, $P' = (\frac{\pi}{2}, \frac{\pi}{2}, 0)$, $Q = (\frac{\pi}{2}, 0, 0)$, $Z = P + Q$.

trace. Indeed for instance $*\,\pi(\ell(\xi))\,e_{123}$ is a linear combination of e_{356} and e_{456} which are orthogonal to e_{123}.
0n the 8 dimensional subspace of e_{ijk}, $i \in \{1,2\}$, $j \in \{3,4\}$, $k \in \{5,6\}$ one gets the product of what happens in the 2-dimensional case with a basis e_1, e_2 of $\wedge^1 \mathbb{C}^2$. One has $*\,e_1 = e_2$, $*\,e_2 = -e_1$ and the representation is given by $e_1 \to \cos\xi\, e_1 + \sin\xi\, e_2$, $e_2 \to \cos\xi\, e_2 - \sin\xi\, e_1$ so that the trace of $*\,\pi(\ell(\xi))$ is $-2\sin\xi$. Thus we get,

$$\operatorname{Trace}(*\,\pi(\ell(\xi))) = -\,8\,\sin\xi_1\,\sin\xi_2\,\sin\xi_3\,. \tag{4.4}$$

b) In terms of $SO(6)$ the Weyl group W maps to the permutations of (ψ_1, ψ_2, ψ_3) and the kernel of this map is the Klein subgroup which is given by the transformation,

$$\psi_j \to \varepsilon_j\,\psi_j \qquad \prod_1^3 \varepsilon_j = 1 \qquad (\varepsilon_j \in \{\pm 1\}). \tag{4.5}$$

By construction the function $\chi(\xi)$ being a (virtual) character is invariant by these transformations and so is the flow X.
c) Let us rewrite $X = \sum \frac{\partial\chi}{\partial\psi_j}\,\frac{\partial}{\partial\,\psi_j}$ in terms of the coordinates φ_j. One has

$$d\,\varphi_j = d\,\psi_k + d\,\psi_\ell\,, \qquad \sum \frac{\partial h}{\partial\varphi_j}\,d\,\varphi_j = \sum\left(\frac{\partial h}{\partial\varphi_k} + \frac{\partial h}{\partial\varphi_\ell}\right)d\,\psi_j\,, \qquad \frac{\partial h}{\partial\psi_j} = \frac{\partial h}{\partial\varphi_k} + \frac{\partial h}{\partial\varphi_\ell}$$

and X is given by

$$\begin{aligned} X = \sum\left(\frac{\partial\chi}{\partial\psi_k} + \frac{\partial\chi}{\partial\psi_\ell}\right)\frac{\partial}{\partial\varphi_j} \quad &= \quad 2\sum(\cos 2\,\psi_k\;\; \sin 2\,\psi_\ell + \;\cos 2\,\psi_\ell\,\sin 2\,\psi_k)\,\sin 2\,\psi_j\,\frac{\partial}{\partial\varphi_j} \\ &= \quad 2\sum \sin(2\,\varphi_j)\,\sin(\varphi_k + \varphi_\ell - \varphi_j)\,\frac{\partial}{\partial\varphi_j}\,. \end{aligned}$$

$\square$

4.2. Invariance of $\mathbb{R}^4_\varphi$ under the flow F.

We let

$$C_+ = \{(\varphi,\varphi,0)\}\,, \quad C_- = \{(\frac{\pi}{2},\varphi,\varphi + \frac{\pi}{2}\}\,. \tag{4.6}$$

The critical set of the flow X is described as follows in terms of the action of the Weyl group W,

Lemma 4.2. *The critical set C of X is given by*

$$C = W(C_+) \cup W(C_-) \cup W(P)\,, \qquad P = \left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right)\,.$$

One checks this directly in the ψ variables.
In order to perform a change of variables we use the function,

$$\delta(\varphi) = \prod \sin\varphi_j \cos(\varphi_k - \varphi_\ell) \tag{4.7}$$

and we let

$$D = \{\varphi, \delta(\varphi) = 0\}\,. \tag{4.8}$$

By construction D is invariant under the group S_3 of permutations of the φ_j's.

Lemma 4.3. *Let $\varphi \notin C$ then there exists $g \in W$ such that $g\varphi \notin D$.*

Proof. Let us first show that the conclusion holds if one of the φ_j vanishes (we always work modulo π and all equalities below have this meaning).
Thus assume that $\varphi_3 = 0$, *i.e.* that $\varphi = (\varphi_1, \varphi_2, 0)$. Then if $\varphi_1 = \varphi_2$ one is in C thus we can assume that $\varphi_1 \neq \varphi_2$. By the transformation (3.23) we get

$$\varphi' = (-\,\varphi_1, -\,\varphi_1, \varphi_2 - \varphi_1)$$

If $\varphi_1 = 0$ we treat $(0, \varphi_2, 0) \sim (\varphi_2, 0, 0)$ by applying (3.23) which gives $(-\,\varphi_2, -\,\varphi_2, -\,\varphi_2)$ for which $\delta(\varphi)$ is 0 only if $\varphi_2 = 0$ in which case we are dealing with the point $O = (0, 0, 0)$ which is in C.
Thus we can assume $\varphi_1 \neq 0$, $\varphi_2 - \varphi_1 \neq 0$ and we get $\prod \sin\varphi'_j \neq 0$.
One has $\varphi'_k - \varphi'_\ell \in \{0, \pm\varphi_2\}$ thus the product $\prod \cos(\varphi'_k - \varphi'_\ell)$ vanishes only if $\varphi_2 = \frac{\pi}{2}$. We are thus dealing with $\left(\varphi_1, \frac{\pi}{2}, 0\right)$ (with $\varphi_1 \neq 0$). We apply (3.23) to $\left(\frac{\pi}{2}, \varphi_1, 0\right)$ which gives

$$\varphi' = \left(-\frac{\pi}{2}, -\frac{\pi}{2}, \varphi_1 - \frac{\pi}{2}\right)$$

If $\varphi_1 = \frac{\pi}{2}$ then one is in C otherwise $\prod \sin\varphi'_j \neq 0$. One has $\varphi'_k - \varphi'_\ell \in \{0, \pm\varphi_1\}$ and since $\varphi_1 \neq \frac{\pi}{2}$ we get $\prod \cos(\varphi'_k - \varphi'_\ell) \neq 0$
We have thus shown that if $\varphi_j = 0$ for some j and $\varphi \notin C$ we can find $g \in W$ with $g\varphi \notin D$.
Thus we can now assume that all $\varphi_j \neq 0$. Thus $\prod \sin\varphi_j \neq 0$. If $\prod \cos(\varphi_k - \varphi_\ell) = 0$ we can assume $\varphi_1 - \varphi_2 = \frac{\pi}{2}$. We then apply (3.23) and get

$$\varphi' = \left(-\,\varphi_1, \varphi_3 - \varphi_1, -\frac{\pi}{2}\right)$$

If one of the components of φ' is 0 we are back to the previous case thus we can assume $\prod \sin\varphi'_j \neq 0$. The $\varphi'_k - \varphi'_\ell$ give (up to sign) φ_3, $\varphi_3 - \varphi_1 + \frac{\pi}{2}$, $\varphi_1 - \frac{\pi}{2}$ and since $\varphi_1 \neq 0$ and $\varphi'_2 = \varphi_3 - \varphi_1 \neq 0$, $\prod \cos(\varphi'_k - \varphi'_\ell) = 0$ can occur only if $\cos\varphi_3 = 0$. In that case the original φ is $\left(\varphi_1, \varphi_1 - \frac{\pi}{2}, \frac{\pi}{2}\right)$ which is in C. $\square$

Note then that one can always choose the $g \in W$ in the Klein subgroup $H \subset W$. Indeed H is a normal subgroup and $S_3 \subset W$ acts as the group of permutation of the φ's and preserves D. Thus for $g = \sigma\, k_1$, $\sigma \in S_3$, $\sigma\, k_1\, \varphi \notin D \Rightarrow k_1\, \varphi \notin D$.

Let us now use this lemma to simplify the presentation of the algebra.

Lemma 4.4. *If $\delta(\varphi) \neq 0$, there exists 4 non zero scalars $\in i^N\, \mathbb{R}^*$, such that one has*

$$(\sin\varphi_k)\, \lambda_\ell\, \lambda_m + \cos(\varphi_\ell - \varphi_m)\, \lambda_0\, \lambda_k = 0$$

and $\prod \lambda_\mu = -\delta(\varphi)$.

Proof. Let us choose the square roots

$$\lambda_0 = (\prod \sin\varphi_j)^{1/2}\,, \quad \lambda_k = \left(\sin\varphi_k \prod_{\ell \neq k} \cos(\varphi_k - \varphi_\ell)\right)^{1/2}$$

in such a way that $\prod\, \lambda_j = -\delta(\varphi)$. Note indeed that the product of the squares gives

$$\prod (\sin\varphi_j)^2 \prod_{k<\ell} (\cos(\varphi_k - \varphi_\ell))^2 = \delta(\varphi)^2$$

and thus one can fix the λ_k and then choose the sign of λ_0 so that it fits. Then to prove the 3 equalities one can multiply by $\lambda_0\,\lambda_k$ which gives $(\sin\varphi_k)(-\delta(\varphi))+\cos(\varphi_\ell-\varphi_m)\,\lambda_0^2\,\lambda_k^2$. Thus one needs to check

$$\sin\varphi_k\prod\cos(\varphi_{\ell_1}-\varphi_{m_1})=\cos(\varphi_\ell-\varphi_m)\prod_{\ell_1\neq k}\cos(\varphi_k-\varphi_{\ell_1})\sin\varphi_k$$

which is an identity. □

One then lets

$$S_\mu=\lambda_\mu\,x^\mu\,,(\text{no summation on }\mu). \tag{4.9}$$

Multiplying (2.24) by $\lambda_0\,\lambda_1\,\lambda_2\,\lambda_3$ one gets

$$[S_\ell,S_m]=i\,[S_0,S_k]_+ \tag{4.10}$$

while (2.25) gives:

$$\cos\varphi_k[S_0,S_k]=i\,\frac{\lambda_0\lambda_k}{\lambda_\ell\lambda_m}\sin(\varphi_\ell-\varphi_m)\,[S_\ell,S_m]_+\,. \tag{4.11}$$

Note that

$$\frac{\lambda_0\lambda_k}{\lambda_\ell\lambda_m}=\frac{\prod\lambda_\mu}{\lambda_\ell^2\lambda_m^2}=\frac{-\delta(\varphi)}{\lambda_\ell^2\lambda_m^2}$$

Proposition 4.5. *If $\delta(\varphi)\neq 0$ the quadratic algebra of $\mathbb{R}^4_\varphi$ admits the presentation given by* (4.10) *and*

$$[S_0,S_k]=i\,J_{\ell m}[S_\ell,S_m]_+ \tag{4.12}$$

where the $J_{\ell m}$ are given by

$$J_{\ell m}=-\tan\varphi_k\tan(\varphi_\ell-\varphi_m)\,. \tag{4.13}$$

If $\varphi_k=\frac{\pi}{2}$ and $\varphi_\ell=\varphi_m$ the corresponding relation gives $0=0$. If $\varphi_k=\frac{\pi}{2}$ and $\varphi_\ell\neq\varphi_m$ it gives $[S_\ell,S_m]_+=0$.

Proof. One needs to show that

$$\frac{-\delta(\varphi)}{\lambda_\ell^2\lambda_m^2}\,\frac{\sin(\varphi_\ell-\varphi_m)}{\cos\varphi_k}=-\tan\varphi_k\tan(\varphi_\ell-\varphi_m)$$

This amounts to the equality

$$\sin\varphi_\ell\,\sin\varphi_m\cos^2(\varphi_\ell-\varphi_m)\cos(\varphi_\ell-\varphi_k)\cos(\varphi_m-\varphi_k)=\lambda_\ell^2\,\lambda_m^2$$

which follows from the definition of the λ_k's. □

We now find a better way to write the $J_{\ell m}$. We first pass in a faithful manner from the φ_j to

$$t_j=\tan\varphi_j\in\mathbb{R}\qquad(\text{recall }\varphi_j\neq\frac{\pi}{2})\,. \tag{4.14}$$

Note also that $t_j\neq 0$ since we are on $\delta(\varphi)\neq 0$.

We then introduce the following new parameters,

$$s_k(\varphi)=1+t_\ell\,t_m\,. \tag{4.15}$$

Lemma 4.6. a) *The map* $\varphi \xrightarrow{\sigma} s = (s_k(\varphi))$ *on* $\Omega = \{\varphi \mid \cos(\varphi_k) \neq 0 \;\forall\, k\,, \delta(\varphi) \neq 0\}$ *is a double cover of the open subset of* $\mathbb{R}^3$

$$\prod s_k \neq 0 \quad \text{and} \quad \prod (s_k - 1) > 0\,. \tag{4.16}$$

b) *The map* σ *is a diffeomorphism of the interior* A° *of* A *with* $\sigma(A^\circ) = \{s \mid 1 < s_1 < s_2 < s_3\}$.

c) *The map* σ *is a diffeomorphism of* B° *with* $\sigma(B^\circ) = \{s \mid s_3 < s_2 < 0, 1 < s_1\}$.

Proof. a) The condition $\delta(\varphi) \neq 0$ shows that $s_k \neq 0\,,\ \forall\, k$. Indeed $\tan\varphi_\ell \tan\varphi_m = -1$ means $\cos(\varphi_\ell - \varphi_m) = 0$. By construction $s_k \neq 1$ and $\prod(s_k - 1) = \prod t_\ell^2 > 0$. Knowing the s_k's one gets $(\prod t_k)^2 = \prod(s_k - 1)$ and choosing the sign of the square root gives $p = \prod t_k$ and then $t_k = p(s_k - 1)^{-1}$. Thus one gets a double cover and the range is characterized by the conditions (4.16). The deck transformation is simply $\varphi \to -\varphi$.

b) On $A^\circ = \{\varphi \mid \frac{\pi}{2} > \varphi_1 > \varphi_2 > \varphi_3 > 0\}$ one has $t_k > 0$ and thus $\prod t_k > 0$ so that the above map is one to one. One checks that the range $\sigma(A^\circ)$ is given by $\sigma(A^\circ) = \{s \mid 1 < s_1 < s_2 < s_3\}$.

c) On $B^\circ = \{\varphi \mid \frac{\pi}{2} + \varphi_3 > \varphi_1 > \frac{\pi}{2} > \varphi_2 > \varphi_3\}$ one has $t_1 < 0, t_2 > 0, t_3 > 0$ and thus $\prod t_k < 0$ so that the above map is one to one. One checks that the range $\sigma(B^\circ)$ is given by $\sigma(B^\circ) = \{s \mid s_3 < s_2 < 0, 1 < s_1\}$. ∎

We define the transformation ρ by

$$\rho(s)_k = \frac{s_\ell - s_m}{s_k}$$

Lemma 4.7. a) *Let* $s_k \neq 0$, *then one has* $\prod(1 + \rho(s)_k) = \prod(1 - \rho(s)_k)$.

b) *If* $\rho(s) = \rho(s')$ *there exists* $\lambda \neq 0$ *with* $s' = \lambda\, s$.

c) *The transformation* $s \to \tilde{s}$,

$$\tilde{s}_k = \frac{-s_k + s_\ell + s_m}{s_\ell s_m}$$

is involutive and $\rho(\tilde{s}) = -\rho(s)$.

Proof. a) Both products give, up to the denominator $\prod s_k$, the product of $(s_k + s_\ell - s_m)$.

b) Consider the 3 linear equations (for fixed ρ_k) given by

$$s_\ell - s_m - \rho_k\, s_k = 0\,. \tag{4.17}$$

This corresponds to the 3×3 matrix given by

$$M(\rho) = \begin{bmatrix} \rho_1 & -1 & 1 \\ 1 & \rho_2 & -1 \\ -1 & 1 & \rho_3 \end{bmatrix}. \tag{4.18}$$

The condition a) means exactly that $\mathrm{Det}\,(M(\rho)) = 0$. Moreover we claim that the rank of $M(\rho)$ is 2. Indeed ρ_2 appears in the two minors $\begin{bmatrix} 1 & \rho_2 \\ -1 & 1 \end{bmatrix}$ which gives $1 + \rho_2$ and $\begin{bmatrix} -1 & 1 \\ \rho_2 & -1 \end{bmatrix}$ which gives $1 - \rho_2$ and one of them is $\neq 0$.

This shows that the kernel is 1-dimensional and hence $\rho(s) = \rho(s')$ implies $s' = \lambda\, s$ for some λ.

c) The relation between s and $\tilde{s}$ can be written as

$$s_\ell\, \tilde{s}_m + s_m\, \tilde{s}_\ell = 2\,. \tag{4.19}$$

The determinant of the system is $2\,s_1\,s_2\,s_3$ so that (4.19) determines $\tilde{s}_k$ uniquely. The relation is clearly symmetric. Finally

$$\rho(\tilde{s})_1 = \frac{\tilde{s}_2 - \tilde{s}_3}{\tilde{s}_1} = \frac{(-\,s_2 + s_1 + s_3)\,s_2 - (-\,s_3 + s_1 + s_2)\,s_3}{s_1\,(-\,s_1 + s_2 + s_3)} = \frac{s_3 - s_2}{s_1} = -\rho(s)_1\,.$$

□

We now get

$$\rho \circ s\,(\varphi)_k = J_{\ell m} \qquad (\varphi_k \neq \frac{\pi}{2} \;\text{ and } \delta(\varphi) \neq 0)\,. \tag{4.20}$$

Indeed

$$\frac{s_\ell - s_m}{s_k} = \frac{t_k t_m - t_k t_\ell}{1 + t_\ell t_m} = \tan\varphi_k \tan(\varphi_m - \varphi_\ell) = J_{\ell m}\,.$$

Lemma 4.8. 1) *The flow X fulfills*

$$X\,s_k(\varphi) = 4 \prod \sin\varphi_j \;\; s_k(\varphi)\,. \tag{4.21}$$

2) *Let $\varphi, \varphi' \in A$ with $\varphi_3 > 0$, $\varphi'_3 > 0$. The following conditions are equivalent:*

a) $J_{\ell m}(\varphi') = J_{\ell m}(\varphi)\,, \quad \forall k$

b) *φ' belongs to the orbit of φ by the flow X.*

3) *The same statement holds for $\varphi, \varphi' \in B$ with $\varphi_3 + \frac{\pi}{2} > \varphi_1$, $\varphi'_3 + \frac{\pi}{2} > \varphi'_1$.*

Proof. 1) One has

$$X(t_k) = \sin(2\,\varphi_k)\sin(-\,\varphi_k + \varphi_\ell + \varphi_n)\,\frac{\partial}{\partial\varphi_k}\tan\varphi_k = 2\tan\varphi_k\sin(-\,\varphi_k + \varphi_\ell + \varphi_m)\,,$$

$$X(s_k) = X(t_\ell)\,t_m + t_\ell\,X(t_m) = 2\,t_\ell\,t_m(\sin(-\,\varphi_\ell + \varphi_m + \varphi_k) + \sin(-\,\varphi_m + \varphi_\ell + \varphi_k))$$

and using $\sin(a+b) + \sin(a-b) = 2\sin a\cos b$, one gets

$$X(s_k) = 4\,t_\ell\,t_m(\sin\varphi_k\cos(\varphi_m - \varphi_\ell)) = 4\,\left(\prod \sin\varphi_j\right)\,\frac{\cos(\varphi_m - \varphi_\ell)}{\cos\varphi_m\cos\varphi_\ell} = 4\left(\prod \sin\varphi_j\right)(1 + t_m\,t_\ell)$$

2) In fact 1) shows that the flow X is up to a non-zero change of speed the scaling flow in $\sigma(A)$. By construction $J_{\ell m}$ has homogeneity degree zero in s_k thus it is preserved by X and b) $\Rightarrow$ a).
Let us show that a) $\Rightarrow$ b). We assume first that $\frac{\pi}{2} > \varphi_1$. The same then holds for φ' uisng a). By Lemma 4.7 the equality a) implies that $\sigma(\varphi') = \lambda\,\sigma(\varphi)$ for some non-zero scalar λ. By Lemma 4.6 the image $\sigma(A^\circ)$ is convex (as well as its closure) and thus the segment $[\sigma(\varphi),\, \sigma(\varphi')]$ is contained in $\sigma(A)$ and its preimage under σ is a segment in a flow line.
On the face Y determined by $\varphi_1 = \frac{\pi}{2}$ one has assuming $\varphi_2 < \frac{\pi}{2}$ the equalities

$$J_{12} = -\tan\varphi_3\tan\left(\frac{\pi}{2} - \varphi_2\right) = -\,t_3/t_2\,, \quad J_{31} = -\tan\varphi_2\tan\left(\varphi_3 - \frac{\pi}{2}\right) = t_2/t_3$$

Moreover one has $X(t_2) = \cos(\varphi_2 - \varphi_3)\,2\,t_2$, $X(t_3) = \cos(\varphi_2 - \varphi_3)\,2\,t_3$ so that the flow X restricts as the scaling flow (up to a non-zero change of speed) in the parameters t_j. Since $J_{23} = \infty$ holds iff $\varphi_1 = \frac{\pi}{2}$ for $\varphi \in A$ we see that a) then implies $\varphi'_1 = \frac{\pi}{2}$ and the proportionality $t'_j = \lambda t_j$. Since the allowed t_j are simply constrained by the inequalities $t_2 \geq t_1 > 0$ the same convexity argument applies. Finally let φ with $\varphi_1 = \varphi_2 = \frac{\pi}{2}$, $\varphi_3 \neq \frac{\pi}{2}$. Then the only remaining parameter is t_3 and one has $X(t_3) = 2\sin\varphi_3\,t_3$. Thus one is dealing with a single flow line.
The proof of 3) is similar. □

5. The geometric data of $\mathbb{R}^4_\varphi$

In this section we compute the geometric data of the quadratic algebras of functions on $\mathbb{R}^4_\varphi$ for all values of the parameter φ. These fall in eleven different classes in each of which one gets further invariants.

5.1. **The definition and explicit matrices.**

We give a list of the characteristic varieties and correspondences. There are 11 different cases. They are described in terms of the φ-coordinates but the result will then be translated in invariant terms using the roots. Let us recall the definition of the geometric data $\{E\,,\,\sigma\,,\,\mathcal{L}\}$ for quadratic algebras. Let $\mathcal{A}=A(V,R)=T(V)/(R)$ be a quadratic algebra where V is a finite-dimensional complex vector space and where (R) is the two-sided ideal of the tensor algebra $T(V)$ of V generated by the subspace R of $V\otimes V$. Consider the subset of $V^*\times V^*$ of pairs (α,β) such that

$$\langle \omega, \alpha\otimes\beta\rangle = 0,\ \ \alpha\neq 0, \beta\neq 0 \tag{5.1}$$

for any $\omega\in R$. Since R is homogeneous, (5.1) defines a subset

$$\Gamma\subset P(V^*)\times P(V^*)$$

where $P(V^*)$ is the complex projective space of one-dimensional complex subspaces of V^*. Let E_1 and E_2 be the first and the second projection of Γ in $P(V^*)$. It is usually assumed that they coincide i.e. that one has

$$E_1=E_2=E\subset P(V^*) \tag{5.2}$$

and that the correspondence σ with graph Γ is an automorphism of E, $\mathcal{L}$ being the pull-back on E of the dual of the tautological line bundle of $P(V^*)$. The algebraic variety E is refered to as the characteristic variety.

The algebras we consider have 4 generators and six relations, thus their characteristic variety is obtained as the locus of points where a 4×6 matrix has rank less than 4. The various matrices depend upon the choice of the parameters and are listed below. We first give the matrix corresponding to the original quadratic algebra of $\mathbb{R}^4_\varphi$,

$$\begin{bmatrix}
-\cos(\varphi_1)\,x_1 & \cos(\varphi_1)\,x_0 & -i\sin(\varphi_2-\varphi_3)\,x_3 & -i\sin(\varphi_2-\varphi_3)\,x_2\\
-\cos(\varphi_2)\,x_2 & i\sin(\varphi_1-\varphi_3)\,x_3 & \cos(\varphi_2)\,x_0 & i\sin(\varphi_1-\varphi_3)\,x_1\\
-\cos(\varphi_3)\,x_3 & -i\sin(\varphi_1-\varphi_2)\,x_2 & -i\sin(\varphi_1-\varphi_2)\,x_1 & \cos(\varphi_3)\,x_0\\
i\sin(\varphi_3)\,x_3 & -\cos(\varphi_1-\varphi_2)\,x_2 & \cos(\varphi_1-\varphi_2)\,x_1 & i\sin(\varphi_3)\,x_0\\
i\sin(\varphi_1)\,x_1 & i\sin(\varphi_1)\,x_0 & -\cos(\varphi_2-\varphi_3)\,x_3 & \cos(\varphi_2-\varphi_3)\,x_2\\
i\sin(\varphi_2)\,x_2 & \cos(\varphi_1-\varphi_3)\,x_3 & i\sin(\varphi_2)\,x_0 & -\cos(\varphi_1-\varphi_3)\,x_1
\end{bmatrix} \tag{5.3}$$

When we pass to the Sklyanin algebra and eliminate the factors i in replacing $Z_0=i\,S_0\,,\quad Z_k=S_k\,,$ we get the following matrix, with $\alpha=-J_{23}$, $\beta=-J_{31}$, $\gamma=-J_{12}$,

$$\begin{bmatrix}
z_1 & -z_0 & \alpha\,z_3 & \alpha\,z_2\\
z_2 & \beta\,z_3 & -z_0 & \beta\,z_1\\
z_3 & \gamma\,z_2 & \gamma\,z_1 & -z_0\\
z_3 & z_2 & -z_1 & z_0\\
z_1 & z_0 & z_3 & -z_2\\
z_2 & -z_3 & z_0 & z_1
\end{bmatrix} \tag{5.4}$$

The fifteen minors of the matrix (5.3) are listed in factorized form in Appendix 1 and those of the matrix (5.4) in subsection 5.3 below (see [27]).

5.2. **The Table.**

We give below the list of the characteristic varieties and correspondences in all cases. Given $z \in \mathbb{C}$ we let $\sigma(z)$ be the conjugacy class of semi-simple automorphisms of a curve of genus 0 with eigenvalues $\{z, z^{-1}\}$. By construction $\sigma(z) = \sigma(z^{-1})$.
The identification of the corresponding algebras $C_{\mathrm{alg}}(\mathbb{R}^4_\varphi)$ in the nongeneric cases (cases 2 to 11 in the table below) are described in section 8 and may be summarized as follows. In case 2, $C_{\mathrm{alg}}(\mathbb{R}^4_\varphi)$ is isomorphic to a homogenized version (quadratic) $U_q(\mathfrak{su}(2))^{\mathrm{hom}}$ of $U_q(\mathfrak{su}(2))$ with either $q \in \mathbb{C}$ with $|q| = 1$ or $q \in \mathbb{R}$ with $0 < q < 1$ or $-1 < q < 0$; it is important to notice that this is a $*$-isomorphism, (that is the $\mathfrak{su}(2)$ does really matter in this notation). Case 3 is obtained by duality from case 2 as explained in Section 7. In case 4, there is a missing relation so $C_{\mathrm{alg}}(\mathbb{R}^4_\varphi)$, which corresponds formally to a version $q = 0$ of $U_q(\mathfrak{su}(2))^{\mathrm{hom}}$, is of exponential growth. Case 5 is obtained by α_3-duality (section 7) from case 6 which is isomorphic to a homogenized version $U(\mathfrak{su}(2))^{\mathrm{hom}}$ of the universal enveloping algebra of $\mathfrak{su}(2)$ (i.e. $q = 1$ in $U_q(\mathfrak{su}(2))^{\mathrm{hom}}$). Case 7 is the θ-deformation studied in Part I [13] and in [17] while case 8 (anti θ-deformation) is obtained by α_1-duality (cf. section 7) from case 7. Case 9 is very singular : 3 relations are missing. Case 10 is obtained by α_3-duality (cf. section 7) from case 11 which is the ordinary algebra of polynomials with 4 indeterminates $\mathbb{C}[x^0, x^1, x^2, x^3]$(classical case). Note that the reality conditions above correspond to the hermiticity of the x^μ and <u>not</u> of the Sklyanin generators S_μ.
It is important to describe the stratification in terms of the roots. The stratas of codimension k correspond to intersections of k singular hyperplanes (up to a factor $\frac{1}{2}$) of the form

$$F((\alpha_1, \cdots, \alpha_k), (n_1, \cdots, n_k)) = \cap \frac{1}{2} H_{(\alpha_j, n_j)}\,, \qquad \alpha_j \in \Delta\,, n_j \in \mathbb{Z}\,.$$

The two dimensional stratas are defined using a single root $\alpha \in \Delta$ (*i.e.* $k = 1$) and since the Weyl group W acts transitively on Δ only the parity of n matters which gives the two kinds of faces F_1 corresponding to n even and F_2 to n odd.
The one dimensional stratas are defined using two roots α, $\beta \in \Delta$ (*i.e.* $k = 2$). The roots only matter up to sign and their relative positions is described by their angle which (up to the sign) can be $\frac{\pi}{2}$ in which case we write $\alpha \perp \beta$ or $\frac{2\pi}{3}$ in which case we write $\alpha - \beta$. We thus have the following one dimensional stratas according to the parity of the n_j.

- $\alpha \perp \beta$ and (n_1, n_2) =(even, odd) gives the line L of case 4 below.
- $\alpha - \beta$ and (n_1, n_2) =(even, odd) or (n_1, n_2) =(odd, odd) gives the line L' of case 5 below.
- $\alpha - \beta$ and (n_1, n_2) =(even, even) gives the line L'' of case 6 below.
- $\alpha \perp \beta$ and (n_1, n_2) =(even, even) gives the line C_+ of case 7 below.
- $\alpha \perp \beta$ and (n_1, n_2) =(odd, odd) gives the line C_- of case 8 below.

To double check that the list is complete[4] one can use Lemma 4.2 to control the critical set C and then assume by Lemma 4.3 that $\delta(\varphi) \neq 0$. Then if $\varphi \in H_{(\alpha,n)}$ and n is even (resp. odd) the root α is one of the differences $\varphi_k - \varphi_l$ (resp. φ_k). Thus up to permutations of the φ_k one obtains one of the cases 1)-6). The complete table giving the geometric datas is the following :

[4]we are grateful to Marc Bellon for pointing out the subtelty of case 9) which was incomplete in an earlier version

Case	**Point in** $\mathcal{M}$	**Characteristic variety**	**Correspondence**
1 Generic	$\alpha(\varphi) \notin \frac{\pi}{2}\mathbb{Z}$	4 points $\cup$ Elliptic curve	$(\mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id},$ translation$)$
2 Even Face	$\varphi_1 = \varphi_2\,, \varphi_2 - \varphi_3 \notin \frac{\pi}{2}\mathbb{Z}$ $\varphi_j \notin \frac{\pi}{2}\mathbb{Z}$,	2 points, 1 line, 2 conics	$\left(\mathrm{id}, \mathrm{id}, \mathrm{id}, \sigma(\frac{i+\alpha^{1/2}}{i-\alpha^{1/2}}), \sigma(\frac{i+\alpha^{1/2}}{i-\alpha^{1/2}})\right)$ $\alpha = -J_{23}$
3 Odd Face	$\varphi_1 = \frac{\pi}{2}$, $\varphi_2 - \varphi_3 \notin \frac{\pi}{2}\mathbb{Z}$ $\varphi_k \notin \frac{\pi}{2}\mathbb{Z}$, $k = 2, 3$	2 points, 1 line, 2 conics	$\Big(\mathrm{id}, \mathrm{id}, -\mathrm{id},$ exchange with square $\sigma(\frac{i+\beta^{1/2}}{i-\beta^{1/2}})^2$, $\beta = -J_{31}\Big)$
4 $\alpha \perp \beta$ (0,1)	$L = \{\left(\frac{\pi}{2}, \varphi, \varphi\right)\}$	six lines	$(\mathrm{id}, -\mathrm{id},$ cyclic permutation of 4 lines (iso, coarse, iso, coarse))
5 $\alpha - \beta$ (0,1)	$L' = \{\left(\frac{\pi}{2}, \frac{\pi}{2}, \varphi\right)\}$	point $\cup\, P_2(\mathbb{C})$	(id, Symmetry of determinant -1)
6 $\alpha - \beta$ (0,0)	$L'' = \{(\varphi, \varphi, \varphi)\}$	point $\cup\, P_2(\mathbb{C})$	(id, id)
7 $\alpha \perp \beta$ (0,0)	$C_+ = \{(\varphi, \varphi, 0)\}$	six lines	$(\mathrm{id}, \mathrm{id}, \sigma(e^{\pm 2i\varphi}))$
8 $\alpha \perp \beta$ (1,1)	$C_- = \{\left(\varphi + \frac{\pi}{2}, \frac{\pi}{2}, \varphi\right)\}$	six lines	$(-\mathrm{id}, -\mathrm{id},$ exchanges with square $\sigma(e^{\pm 4i\varphi}))$
9	$P = \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$	$P_3(\mathbb{C})$	Symmetry of determinant -1 and point $\rightarrow$ line on a quadric
10	$P' = \left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$ (in $C_+ \cap C_-$)	$P_3(\mathbb{C})$	Symmetry of determinant 1
11	$0 = (0, 0, 0)$ (in C_+)	$P_3(\mathbb{C})$	id

(5.5) **The Geometric Data**

5.3. **Generic case.**

We now give the detailed description of the geometric data starting with the generic case. This case is defined by

$$G = \left\{\varphi ;\ \ \varphi_j \notin \frac{\pi}{2}\,\mathbb{Z}\,,\ \varphi_k - \varphi_\ell \notin \frac{\pi}{2}\,\mathbb{Z}\right\}. \tag{5.6}$$

Then the $J_{k\ell}$ are well defined and are $\neq 0$. Let us show that we cannot have $J_{12} = 1$, $J_{23} = -1$. Indeed $\tan\varphi_3 \tan(\varphi_1 - \varphi_2) = -1$ means $\pi/2 - \varphi_3 = \varphi_2 - \varphi_1$ while $\tan\varphi_1 \tan(\varphi_2 - \varphi_3) = 1$ means $\frac{\pi}{2} - \varphi_1 = \varphi_2 - \varphi_3$. This gives $\pi - \varphi_1 - \varphi_3 = 2\,\varphi_2 - \varphi_1 - \varphi_3$ and $2\,\varphi_2 = \pi$ which is not allowed by (5.6). We can thus apply the result of Smith-Stafford [27] and get:

Proposition 5.1. *For $\varphi \in G$ the geometric data of $\mathbb{R}^4_\varphi$ is given by 4 points and a non degenerate elliptic curve E, and σ is identity on the 4 points and a translation of E given explicitely in terms of the parameters $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\alpha_3 = \gamma$ with*

$$\alpha_k = -\,J_{\ell m}\,. \tag{5.7}$$

by

$$E = \{z\,;\ \sum_0^3 z_j^2 = 0\,,\ \frac{1-\gamma}{1+\alpha}\,z_1^2 + \frac{1+\gamma}{1-\beta}\,z_2^2 + z_3^2 = 0\}\,. \tag{5.8}$$

and

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} \xrightarrow{\sigma} \begin{bmatrix} -\,2\,\alpha\,\beta\,\gamma\,z_1\,z_2\,z_3 - z_0(-\,z_0^2 + \beta\,\gamma\,z_1^2 + \alpha\,\gamma\,z_2^2 + \alpha\,\beta\,z_3^2) \\ 2\,\alpha\,z_0\,z_2\,z_3 + z_1(z_0^2 - \beta\,\gamma\,z_1^2 + \alpha\,\gamma\,z_2^2 + \alpha\,\beta\,z_3^2) \\ 2\,\beta\,z_0\,z_1\,z_3 + z_2(z_0^2 + \beta\,\gamma\,z_1^2 - \alpha\,\gamma\,z_2^2 + \alpha\,\beta\,z_3^2) \\ 2\,\gamma\,z_0\,z_1\,z_2 + z_3(z_0^2 + \beta\,\gamma\,z_1^2 + \alpha\,\gamma\,z_2^2 - \alpha\,\beta\,z_3^2) \end{bmatrix}. \tag{5.9}$$

Proof. We rely on Smith-Stafford [27]. The first point is to rewrite the algebra of proposition (4.5) in the form,

$$[T_0, T_k]_+ = [T_\ell, T_m]\,,\ [T_0, T_k] = \alpha_k [T_\ell, T_m]_+ \tag{5.10}$$

One simply lets $T_0 = i\,S_0$, $T_k = S_k$. Then (4.10) means $[T_\ell, T_m] = [T_0, T_k]_+$ and (4.12) means $[T_0, T_k] = -\,J_{\ell m}[T_1, T_m]_+$.

Note the crucial $-$ sign in (5.7). By hypothesis on $\varphi \in G$ one has $\alpha\,\beta\,\gamma \neq 0$. Moreover we have seen above that for $\varphi \in G$ one cannot have $J_{12} = 1$, $J_{23} = -1$, *i.e.* $\alpha = -1$, $\beta = 1$ (or any cyclic transformed of that). Now in case one of the α's, say α belongs to ± 1 the equality $\alpha + \beta + \gamma + \alpha\,\beta\,\gamma = 0$, *i.e.* $\prod(1+\alpha) = \prod(1-\alpha)$ shows that both $+1$ and -1 occur and since $(-1, 1, x)$ is excluded the only remaining case is $(1, -1, \gamma)$ with $\gamma \notin \{-1, 0, 1\}$.

Thus the hypothesis of Smith-Stafford are fulfilled and one gets from [27] that besides the 4 points $(1, 0, 0, 0) \ldots$ the characteristic variety is the curve in $P_3(\mathbb{C})$ with equations (5.8). The translation σ being given explicity by (5.9). $\square$

It will be useful for the computations in the degenerate case to display the list of the 15 minors in the case of the Sklyanin algebra, in the above parameters (α,β,γ). Their list is given below ([27]).

$$\begin{gathered}
\{-2\,(\gamma\,z_1\,z_2 - z_0\,z_3)\,(z_0^2+z_1^2+z_2^2+z_3^2),\\
2\,(z_0\,z_2-\beta\,z_1\,z_3)\,(z_0^2+z_1^2+z_2^2+z_3^2),\\
-2\,(z_0\,z_2+z_1\,z_3)\,(z_0^2-\gamma\,(\beta\,z_1^2+z_2^2)+\beta\,z_3^2),\\
2\,(z_0^2\,(-(1+\gamma)\,z_2^2+(-1+\beta)\,z_3^2)+z_1^2\,((-1+\beta)\,\gamma\,z_2^2+\beta\,(1+\gamma)\,z_3^2)),\\
2\,(-z_1\,z_2+z_0\,z_3)\,(z_0^2-\gamma\,(\beta\,z_1^2+z_2^2)+\beta\,z_3^2),\\
2\,(z_0\,z_1-\alpha\,z_2\,z_3)\,(z_0^2+z_1^2+z_2^2+z_3^2),\\
-2\,(z_0\,z_1-z_2\,z_3)\,(z_0^2+\gamma\,z_1^2-\alpha\,(\gamma\,z_2^2+z_3^2)),\\
-2\,(z_1\,z_2+z_0\,z_3)\,(z_0^2+\gamma\,z_1^2-\alpha\,(\gamma\,z_2^2+z_3^2)),\\
2\,(z_0^2\,((-1+\gamma)\,z_1^2-(1+\alpha)\,z_3^2)+z_2^2\,((1+\alpha)\,\gamma\,z_1^2+\alpha\,(-1+\gamma)\,z_3^2)),\\
2\,(z_0^2\,((1+\beta)\,z_1^2-(-1+\alpha)\,z_2^2)-((-1+\alpha)\,\beta\,z_1^2+\alpha\,(1+\beta)\,z_2^2)\,z_3^2),\\
-2\,(z_0\,z_2-z_1\,z_3)\,(z_0^2+\alpha\,z_2^2-\beta\,(z_1^2+\alpha\,z_3^2)),\\
-2\,(z_0\,z_1+z_2\,z_3)\,(z_0^2+\alpha\,z_2^2-\beta\,(z_1^2+\alpha\,z_3^2)),\\
2\,(z_0\,z_2+\beta\,z_1\,z_3)\,(z_0^2+\gamma\,z_1^2-\alpha\,(\gamma\,z_2^2+z_3^2)),\\
2\,(z_0\,z_1+\alpha\,z_2\,z_3)\,(z_0^2-\gamma\,(\beta\,z_1^2+z_2^2)+\beta\,z_3^2),\\
2\,(\gamma\,z_1\,z_2+z_0\,z_3)\,(z_0^2+\alpha\,z_2^2-\beta\,(z_1^2+\alpha\,z_3^2))\}
\end{gathered}\tag{5.11}$$

5.4. **Face $\alpha = n$ and n even.** $F_1 = \{(\varphi_1,\varphi_1,\varphi_3)\}$.

In that case we have a Sklyanin algebra and with the above notations the parameters are $\gamma=0$ while $\beta=-\alpha=\tan\varphi_1\tan(\varphi_3-\varphi_1)$. The list of minors then simplifies as follows,

$$\begin{gathered}
\{z_0\,z_3\,(z_0^2+z_1^2+z_2^2+z_3^2),(z_0\,z_2+\alpha\,z_1\,z_3)\,(z_0^2+z_1^2+z_2^2+z_3^2),\\
-(z_0\,z_2+z_1\,z_3)\,(z_0^2-\alpha\,z_3^2),-\alpha\,z_1^2\,z_3^2-z_0^2\,(z_2^2+(1+\alpha)\,z_3^2),\\
(-z_1\,z_2+z_0\,z_3)\,(z_0^2-\alpha\,z_3^2),(z_0\,z_1-\alpha\,z_2\,z_3)\,(z_0^2+z_1^2+z_2^2+z_3^2),\\
-(z_0\,z_1-z_2\,z_3)\,(z_0^2-\alpha\,z_3^2),-(z_1\,z_2+z_0\,z_3)\,(z_0^2-\alpha\,z_3^2),\\
-\alpha\,z_2^2\,z_3^2-z_0^2\,(z_1^2+(1+\alpha)\,z_3^2),(-1+\alpha)\,(z_1^2+z_2^2)\,(-z_0^2+\alpha\,z_3^2),\\
-(z_0\,z_2-z_1\,z_3)\,(z_0^2+\alpha\,(z_1^2+z_2^2+\alpha\,z_3^2)),-(z_0\,z_1+z_2\,z_3)\,(z_0^2+\alpha\,(z_1^2+z_2^2+\alpha\,z_3^2)),\\
(z_0\,z_2-\alpha\,z_1\,z_3)\,(z_0^2-\alpha\,z_3^2),(z_0\,z_1+\alpha\,z_2\,z_3)\,(z_0^2-\alpha\,z_3^2),\\
z_0\,z_3\,(z_0^2+\alpha\,(z_1^2+z_2^2+\alpha\,z_3^2))\}
\end{gathered}\tag{5.12}$$

The detailed analysis shows that the characteristic variety is the union of the two points with coordinates $(z_0,z_1,z_2,z_3)=(1,0,0,0)$, $(z_0,z_1,z_2,z_3)=(0,0,0,1)$, of the line $\{(0,z_1,z_2,0)\}$ and of the two conics obtained by intersecting the quadric $(z_0^2+z_1^2+z_2^2+z_3^2)=0$ with the two hyperplanes $(z_0^2-\alpha\,z_3^2)=0$.
The correspondence σ is the identity on the two points, and on the line. It restricts to the two conics and is a rational automorphism of each. It admits two fixed points on each of them and its derivative at the fixed points is given by the following complex numbers

$$\frac{i-\sqrt{\alpha}}{i+\sqrt{\alpha}}\,,\quad \frac{i+\sqrt{\alpha}}{i-\sqrt{\alpha}}\,.$$

5.5. **Face $\alpha = n$ and n odd.** $F_2 = \{\left(\frac{\pi}{2},\varphi_2,\varphi_3\right)\}$.

In that case we have a limiting case of the Sklyanin algebra where the parameter $\alpha = \infty$ while $\gamma = -\frac{1}{\beta}$. The list of minors can be computed directly and gives the following, up to non-zero scalar factors,

(5.13)
$$\begin{gathered}\{(z_1\,z_2+\beta\,z_0\,z_3)\,(z_0^2+z_1^2+z_2^2+z_3^2),(z_0\,z_2-\beta\,z_1\,z_3)\,(z_0^2+z_1^2+z_2^2+z_3^2),\\ (-z_0\,z_2-z_1\,z_3)\,(\beta\,(z_0^2+z_1^2)+z_2^2+\beta^2\,z_3^2),(-1+\beta)\,(z_0^2+z_1^2)\,(-z_2^2+\beta\,z_3^2),\\ (-z_1\,z_2+z_0\,z_3)\,(\beta\,(z_0^2+z_1^2)+z_2^2+\beta^2\,z_3^2),-z_2\,z_3\,(z_0^2+z_1^2+z_2^2+z_3^2),(z_0\,z_1-z_2\,z_3)\,(-z_2^2+\beta\,z_3^2),\\ (z_1\,z_2+z_0\,z_3)\,(-z_2^2+\beta\,z_3^2),-\beta\,z_0^2\,z_3^2-z_2^2\,(z_1^2+(1+\beta)\,z_3^2),-\beta\,z_1^2\,z_3^2-z_2^2\,(z_0^2+(1+\beta)\,z_3^2),\\ (-z_0\,z_2+z_1\,z_3)\,(z_2^2-\beta\,z_3^2),(-z_0\,z_1-z_2\,z_3)\,(z_2^2-\beta\,z_3^2),(z_0\,z_2+\beta\,z_1\,z_3)\,(z_2^2-\beta\,z_3^2),\\ z_2\,z_3\,(\beta\,(z_0^2+z_1^2)+z_2^2+\beta^2\,z_3^2),(-z_1\,z_2+\beta\,z_0\,z_3)\,(z_2^2-\beta\,z_3^2)\}\end{gathered}$$

The detailed analysis shows that the characteristic variety is the union of the two points with coordinates $(z_0, z_1, z_2, z_3) = (0, 0, 1, 0)$, $(z_0, z_1, z_2, z_3) = (0, 0, 0, 1)$, of the line $\{(z_0, z_1, 0, 0)\}$ and of the two conics obtained by intersecting the quadric $(z_0^2 + z_1^2 + z_2^2 + z_3^2) = 0$ with the two hyperplanes $(z_2^2 - \beta\, z_3^2) = 0$.

The correspondence σ is the identity on the two points, and is given on the line by

$$\sigma(z_0, z_1, 0, 0) = (z_0, -z_1, 0, 0)\,.$$

It exchanges the two conics and is a rational isomorphism of one with the other, moreover the square of σ admits two fixed points on each of them and its derivative at the fixed points is given by the squares of the following complex numbers

$$\frac{i+\sqrt{\beta}}{i-\sqrt{\beta}}\,, \quad \frac{i-\sqrt{\beta}}{i+\sqrt{\beta}}\,.$$

5.6. Edge $\alpha \perp \beta$ and (n_1, n_2) =(even, odd). $L = \{(\frac{\pi}{2}, \varphi, \varphi)\}$.

From now on we no longer use the change of variables to the Sklyanin algebras but we rely directly on the explicit form of the minors of the original matrix as computed in the Appendix 14. Note in particular that the parameters x_j are no longer the same as the above z_j but this is irrelevant since we compute intrinsic invariants of the quadratic algebra. In the case at hand the list of minors simplifies (with non zero scale factors removed) to the following,

$$\begin{gathered}\{(x_1\,x_2+i\,x_0\,x_3)\,(-x_0^2+x_1^2+x_2^2+x_3^2),\\ (x_0\,x_2+i\,x_1\,x_3)\,(x_0^2-x_1^2-x_2^2-x_3^2),\\ (x_0\,x_2+i\,x_1\,x_3)\,(x_0^2-x_1^2+x_2^2+x_3^2),\ (x_0^2-x_1^2)\,(x_2^2+x_3^2),\\ (-i\,x_1\,x_2+x_0\,x_3)\,(-x_0^2+x_1^2-x_2^2-x_3^2),0,0,0,0,0,0,0,\\ 0,0,0\}\end{gathered} \tag{5.14}$$

Thus the characteristic variety consists in the six lines ℓ_j given in terms of free parameters x_j by

$$\begin{gathered}\{(0,0,x_2,x_3)\}\,,\ \{(x_0,x_1,0,0)\}\,,\ \{(x_0,x_0,x_2,-ix_2)\}\,,\ \{(x_0,x_0,x_2,ix_2)\}\,,\\ \{(x_0,-x_0,x_2,ix_2)\}\,,\ \{(x_0,-x_0,x_2,-ix_2)\}\,.\end{gathered} \tag{5.15}$$

The correspondence σ is the identity on the first line, -1 on the second and permutes cyclically the four others ℓ_j. Passing from ℓ_4 to ℓ_5 or from ℓ_6 to ℓ_3 one gets the coarse correspondence, while the other maps are rational isomorphisms. The following three lines meet and their point of intersection is mapped by the coarse correspondence to the indicated line

$$\ell_1\cap\ell_4\cap\ell_5\to\ell_5\,,\quad \ell_1\cap\ell_3\cap\ell_6\to\ell_3\,,\quad \ell_2\cap\ell_3\cap\ell_4\to\ell_5\,,\quad \ell_2\cap\ell_5\cap\ell_6\to\ell_3\,.$$

This is coherent as the restriction of the relevant coarse correspondence.

5.7. **Edge $\alpha - \beta$ and $(n_1, n_2) =$ (odd, odd) or (even, odd).** $L' = \{(\frac{\pi}{2}, \frac{\pi}{2}, \varphi)\}$.

In that case the list of minors simplifies (with non zero scale factors removed) to the following,

$$
\begin{gathered}
\{\, x_0\, x_3\, (-x_0^2 + x_1^2 + x_2^2 + \sin(\varphi)^2\, x_3^2), \\
x_1\, x_3\, (-x_0^2 + x_1^2 + x_2^2 + \sin(\varphi)^2\, x_3^2),\; x_3^2\, (x_0\, x_2 + i \sin(\varphi)\, x_1\, x_3), \\
(x_0^2 - x_1^2)\, x_3^2,\; x_3^2\, (-i\, x_1\, x_2 + \sin(\varphi)\, x_0\, x_3), \\
x_2\, x_3\, (-x_0^2 + x_1^2 + x_2^2 + \sin(\varphi)^2\, x_3^2),\; x_3^2\, (x_0\, x_1 - i \sin(\varphi)\, x_2\, x_3), \\
x_3^2\, (i\, x_1\, x_2 + \sin(\varphi)\, x_0\, x_3),\; (x_0^2 - x_2^2)\, x_3^2, (x_1^2 + x_2^2)\, x_3^2, \\
x_3^2\, (-x_0\, x_2 + i \sin(\varphi)\, x_1\, x_3), x_3^2\, (x_0\, x_1 + i \sin(\varphi)\, x_2\, x_3), \\
x_1\, x_3^3,\; x_2\, x_3^3,\; x_0\, x_3^3 \}
\end{gathered}
\tag{5.16}
$$

Thus the characteristic variety contains the hyperplane $x_3 = 0$. For $x_3 \neq 0$ the last three minors show that all other coordinates vanish and this gives an additional point, not in the above hyperplane. The correspondence σ is the symmetry

$$\sigma(x_0, x_1, x_2, 0) = (-x_0, x_1, x_2, 0)\,.$$

5.8. **Edge $\alpha - \beta$ and $(n_1, n_2) =$(even, even).** $L'' = \{(\varphi, \varphi, \varphi)\}$.

In that case the list of minors simplifies (with non zero scale factors removed) to the following,

$$
\begin{gathered}
\{\, x_0\, x_3\, (\sin(\varphi)^2\, x_0^2 - x_1^2 - x_2^2 - x_3^2),\; x_0\, x_2\, (\sin(\varphi)^2\, x_0^2 - x_1^2 - x_2^2 - x_3^2), \\
x_0^2\, (\sin(\varphi)\, x_0\, x_2 + i\, x_1\, x_3),\; x_0^2\, (x_2^2 + x_3^2), \\
x_0^2\, (-i\, x_1\, x_2 + \sin(\varphi)\, x_0\, x_3),\; x_0\, x_1\, (\sin(\varphi)^2\, x_0^2 - x_1^2 - x_2^2 - x_3^2), \\
x_0^2\, (\sin(\varphi)\, x_0\, x_1 - i\, x_2\, x_3),\; x_0^2\, (i\, x_1\, x_2 + \sin(\varphi)\, x_0\, x_3), \\
x_0^2\, (x_1^2 + x_3^2),\; x_0^2\, (x_1^2 + x_2^2),\; x_0^2\, (\sin(\varphi)\, x_0\, x_2 - i\, x_1\, x_3), \\
x_0^2\, (\sin(\varphi)\, x_0\, x_1 + i\, x_2\, x_3),\; x_0^3\, x_2,\; x_0^3\, x_1,\; x_0^3\, x_3 \}
\end{gathered}
\tag{5.17}
$$

Thus the characteristic variety contains the hyperplane $x_0 = 0$. For $x_0 \neq 0$ the last three minors show that all other coordinates vanish and this gives an additional point, not in the above hyperplane. The correspondence σ is the identity.

5.9. **Edge $\alpha \perp \beta$ and $(n_1, n_2) =$(even, even).** $C_+ = \{(\varphi, \varphi, 0)\}$.

In that case the list of minors simplifies (with non zero scale factors removed) to the following,

$$
\begin{gathered}
\{\, x_0\, (x_1^2 + x_2^2)\, x_3, (x_1^2 + x_2^2)\, (-i \cos(\varphi)\, x_0\, x_2 + \sin(\varphi)\, x_1\, x_3), \\
(\sin(\varphi)\, x_0\, x_2 + i \cos(\varphi)\, x_1\, x_3)\, (x_0^2 + x_3^2), (x_0^2\, x_2^2 - x_1^2\, x_3^2), \\
x_1\, x_2\, (x_0^2 + x_3^2),\; (x_1^2 + x_2^2)\, (\cos(\varphi)\, x_0\, x_1 - i \sin(\varphi)\, x_2\, x_3), \\
(\sin(\varphi)\, x_0\, x_1 - i \cos(\varphi)\, x_2\, x_3)\, (x_0^2 + x_3^2),\; x_1\, x_2\, (x_0^2 + x_3^2), \\
(x_0^2\, x_1^2 - x_2^2\, x_3^2),\; \sin(4\, \varphi)\, (x_1^2 + x_2^2)\, (x_0^2 + x_3^2), \\
(x_1^2 + x_2^2)\, (i \sin(\varphi)\, x_0\, x_2 + \cos(\varphi)\, x_1\, x_3), \\
(x_1^2 + x_2^2)\, (\sin(\varphi)\, x_0\, x_1 + i \cos(\varphi)\, x_2\, x_3), \\
(\cos(\varphi)\, x_0\, x_2 - i \sin(\varphi)\, x_1\, x_3)\, (x_0^2 + x_3^2), \\
(-i \cos(\varphi)\, x_0\, x_1 + \sin(\varphi)\, x_2\, x_3)\, (x_0^2 + x_3^2), x_0\, (x_1^2 + x_2^2)\, x_3 \}
\end{gathered}
\tag{5.18}
$$

Thus the characteristic variety consists in the six lines ℓ_j given in terms of free parameters x_j by

$$\begin{gathered}\{(0,x_1,x_2,0)\}\,,\ \{(x_0,0,0,x_3)\}\,,\ \{(x_0,x_1,ix_1,ix_0)\}\,,\ \{(x_0,x_1,-ix_1,ix_0)\}\,,\\ \{(x_0,x_1,ix_1,-ix_0)\}\,,\ \{(x_0,x_1,-ix_1,-ix_0)\}\,.\end{gathered}\tag{5.19}$$

The correspondence σ is the identity on the first two lines. It preserves globally the other ℓ_j and induces on each of them the rational automorphism given by multiplication by $e^{\pm 2i\varphi}$.

5.10. **Edge $\alpha \perp \beta$ and (n_1, n_2) =(odd, odd).** $C_- = \{\left(\frac{\pi}{2}+\varphi, \frac{\pi}{2}, \varphi\right)\}$**.**

In that case the list of minors simplifies (with non zero scale factors removed) to the following,

$$\begin{gathered}\{(\sin(\varphi)x_1x_2 + i\cos(\varphi)x_0x_3)(x_0^2-x_2^2),\ x_1x_3(x_0^2-x_2^2),\\ x_0x_2(x_1^2-x_3^2),\ x_0^2x_3^2-x_1^2x_2^2,\\ (i\cos(\varphi)x_1x_2 - \sin(\varphi)x_0x_3)(x_1^2-x_3^2),\\ (-i\sin(\varphi)x_0x_1 + \cos(\varphi)x_2x_3)(x_0^2-x_2^2),\\ (\cos(\varphi)x_0x_1 - i\sin(\varphi)x_2x_3)(x_0^2-x_2^2),\\ (\cos(\varphi)x_1x_2 - i\sin(\varphi)x_0x_3)(x_0^2-x_2^2),\\ \sin(4\varphi)(x_0^2-x_2^2)(x_1^2-x_3^2),\\ (x_0^2x_1^2-x_2^2x_3^2),\ x_0x_2(x_1^2-x_3^2),\\ (\cos(\varphi)x_0x_1 + i\sin(\varphi)x_2x_3)(x_1^2-x_3^2),\ x_1x_3(x_0^2-x_2^2),\\ (i\sin(\varphi)x_0x_1 + \cos(\varphi)x_2x_3)(x_1^2-x_3^2),\\ (\sin(\varphi)x_1x_2 - i\cos(\varphi)x_0x_3)(x_1^2-x_3^2)\}\end{gathered}\tag{5.20}$$

Thus the characteristic variety consists in the six lines ℓ_j given in terms of free parameters x_j by

$$\begin{gathered}\{(0,x_1,0,x_3)\}\,,\ \{(x_0,0,x_2,0)\}\,,\ \{(x_0,x_1,x_0,x_1)\}\,,\ \{(x_0,x_1,-x_0,-x_1)\}\,,\\ \{(x_0,x_1,x_0,-x_1)\}\,,\ \{(x_0,x_1,-x_0,x_1)\}\,.\end{gathered}\tag{5.21}$$

The correspondence σ is -1 on the first two lines. It permutes ℓ_3 with ℓ_4 and its square is the rational automorphism multiplying the ratio x_1/x_0 by $e^{4i\varphi}$. It permutes ℓ_5 with ℓ_6 and its square is the rational automorphism multiplying the ratio x_1/x_0 by $e^{-4i\varphi}$.

5.11. **Vertex** $P = (\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$**.**

In that case all minors vanish identically. Thus the characteristic variety is all projective space. The correspondence σ is given by

$$\sigma((x_0,x_1,x_2,x_3)) = (-x_0,x_1,x_2,x_3)\,,$$

but it degenerates on the quadric

$$\mathcal{Q} = \{x \mid x_0^2 - \sum x_k^2 = 0\}\,,$$

to a correspondence which assigns to any point $p \in \mathcal{Q}$ a line $\ell(p) \subset \mathcal{Q}$ containing the point $\sigma(p)$ and belonging to one of the two families of lines that rule the surface $\mathcal{Q}$.

5.12. **Vertex** $P' = (\frac{\pi}{2}, \frac{\pi}{2}, 0)$**.**

In that case all minors vanish identically. Thus the characteristic variety is all projective space. The correspondence σ is given by

$$\sigma((x_0,x_1,x_2,x_3)) = (-x_0,x_1,-x_2,x_3)\,.$$

5.13. **Vertex** $O = (0,0,0)$**.**

In that case all minors vanish identically and the correspondence σ is the identity.

6. Isomorphism classes of $\mathbb{R}^4_\varphi$ and orbits of the flow F

We let as above $\mathcal{M}$ be the moduli space of oriented non-commutative 3-spheres and $P\mathcal{M}$ its quotient by the symmetry given by proposition 2.4 3). For $\varphi \in \mathcal{M}$ we view the algebra $\mathcal{A} = C_{\rm alg}(\mathbb{R}^4_\varphi)$ as a graded algebra *i.e.* we endow it with the one parameter group of automorphisms which rescale the generators x_ν,

$$\theta_\lambda \in {\rm Aut}\mathcal{A}\,, \quad \theta_\lambda(x_\nu) = \lambda\, x_\nu\,, \quad \forall \lambda \in \mathbb{R}^*\,. \tag{6.1}$$

We let $P\mathcal{M}$ be the quotient of the real moduli space $\mathcal{M}$ by the symmetry of proposition 2.4 3).
This section will be devoted to prove the following result:

Theorem 6.1. *Let $\varphi_j \in \mathcal{M}$ the following conditions are equivalent:*

a) *The graded algebras $C_{\rm alg}(\mathbb{R}^4_{\varphi_j})$ are isomorphic.*
b) *$\varphi_2 \in$ Flow line of φ_1 in $P\mathcal{M}$.*

The proof of *b)* $\Rightarrow$ *a)* was given above in section 4.2. The converse is based on the information given by the geometric data which is by construction an invariant of the graded algebra. The proof will be broken up in the non-generic and generic cases.

6.1. Proof in the non-generic case.

To analyze the information given by the geometric data we can restrict the parameters φ to the fundamental domain $A \cup B$ of proposition 3.7. The symmetry given by proposition 2.4 3) is given explicitely by the transformation

$$\rho(\varphi_1, \varphi_2, \varphi_3) = (\varphi_1, \varphi_1 - \varphi_3, \varphi_1 - \varphi_2) \tag{6.2}$$

The identification γ of the face $(P'QZ)$ with the face (ZPP') (proposition 3.7) followed by the symmetry ρ (6.2) gives the following symmetry $\sigma = \rho \circ \gamma$ of the face $(P'QZ)$,

$$\sigma(\varphi_1, \varphi_2, \varphi_3) = (\pi - \varphi_1 + \varphi_2, \varphi_2, \varphi_2 - \varphi_3) \tag{6.3}$$

For vertices we have three elements 0, P, P' modulo the action of $\Gamma \rtimes W$. The geometric data allows to separate them.
One has a priori nine edges. They fall in five different classes 4) 5) 6) 7) 8).
Using γ and ρ one sees that the following four edges are equivalent:

$$[ZP] \sim [PP'] \sim [QP'] \sim [ZQ]$$

and are all in case 5).
Similarly, using ρ one sees that the following two edges are equivalent:

$$[OP] \sim [OQ]$$

and are both in case 6).
The table of geometric datas shows that the geometric data of $\mathbb{R}^4_\varphi$ determines in which of the five cases 4)-8) one is. Thus when the flow is transitive in the corresponding edge there is nothing to prove. This covers the cases 4) 5) 6). The two cases 7) 8) correspond to fixed points of the flow. For C_+ one gets the edge in $[OP'] \subset \bar{A}$ given by $\{\varphi, \varphi, 0\}$ with $0 < \varphi < \frac{\pi}{2}$. The geometric data gives back the set $e^{\pm 2i\varphi}$ and this allows to recover φ. Thus distinct φ give non-isomorphic quadratic algebras.
For C_- one gets the edge $[P'Z] \subset \bar{B}$ *i.e.* $\{(\frac{\pi}{2} + \varphi, \frac{\pi}{2}, \varphi)\}$ whose interior corresponds to $0 < \varphi < \frac{\pi}{2}$. The geometric data gives back the set $e^{\pm 4i\varphi}$. Thus there is an ambiguity $\varphi \to \frac{\pi}{2} - \varphi$ in φ knowing

the geometric data. To understand it let us note that in fact one checks that σ restricts to the edge $[P'Z]$ as $\varphi \to \frac{\pi}{2} - \varphi$. This then accounts for the above ambiguity.

We now have to deal with the faces. We start with those which are odd (*i.e.* $H_{\alpha,n}$ with n odd). The identification γ of the face $(P'QZ)$ with the face (ZPP') (proposition 3.7) shows that we just need to deal with (ZPP') and with the face (QPP') which is common to A and B.
For the face (ZPP') the equation of the supporting hyperplane is $\varphi_2 = \frac{\pi}{2}$ and one is in case 3) with generic elements of the form $(\varphi_1, \frac{\pi}{2}, \varphi_3)$ where $\varphi_3 + \frac{\pi}{2} > \varphi_1 > \frac{\pi}{2} > \varphi_3$.
The geometric data determines the square

$$\sigma(\frac{i+\beta^{1/2}}{i-\beta^{1/2}})^2$$

where $\beta = -\tan\varphi_3/\tan\varphi_1 > 0$. Then $\frac{i+\beta^{1/2}}{i-\beta^{1/2}}$ is of modulus one and the geometric data determines β up to the ambiguity given by $\beta \to 1/\beta$. But the face (ZPP') admits the symmetry given by

$$\gamma \circ \rho(\varphi_1, \varphi_2, \varphi_3) = (\pi - \varphi_3,\, \pi - \varphi_2,\, \pi - \varphi_1) \tag{6.4}$$

whose effect is precisely the transformation $\beta \to 1/\beta$. Note that the segment joining P to the middle of $[P'Z]$ is globally invariant under the flow X.
For the face (QPP') the equation of the supporting hyperplane is $\varphi_1 = \frac{\pi}{2}$ and one is in case 3) with generic elements of the form $(\frac{\pi}{2},\, \varphi_2,\, \varphi_3)$ where $\frac{\pi}{2} > \varphi_2 > \varphi_3 > 0$. The geometric data determines the square

$$\sigma(\frac{i+\beta^{1/2}}{i-\beta^{1/2}})^2$$

where $\beta = -\tan\varphi_2/\tan\varphi_3 < 0$. Then $\frac{i+\beta^{1/2}}{i-\beta^{1/2}}$ is real and the geometric data determines β up to the ambiguity given by $\beta \to 1/\beta$. But the inequality $\tan\varphi_2 > \tan\varphi_3 > 0$ shows that in fact $\beta \in]-\infty, -1[$ so that the geometric data determines β uniquely.
To summarize we have up to symmetry only two odd faces, the geometric data allows to decide (by $|q| = 1$ or $q \in \mathbb{R}$) in which case one is, and gives back the flow line up to the remaining symmetries.

Let us now consider the even faces (*i.e.* $H_{\alpha,n}$ with n even). Using ρ we get the equivalence $(OPP') \sim (OQP')$. To be able to use lemma 4.8 we concentrate on (OPP') on which $\varphi_3 > 0$. The equation of the supporting hyperplane is $\varphi_1 = \varphi_2$ and one is in case 2) with generic elements of the form $(\varphi_1,\, \varphi_1,\, \varphi_3)$ where $\frac{\pi}{2} > \varphi_1 > \varphi_3 > 0$. The geometric data determines

$$\sigma(\frac{i+\alpha^{1/2}}{i-\alpha^{1/2}})$$

where $\alpha = \tan\varphi_1 \tan(\varphi_1 - \varphi_3) > 0$. Then $q = \frac{i+\alpha^{1/2}}{i-\alpha^{1/2}}$ is of modulus one. Thus the geometric data determines α (since it determines $\alpha^{1/2}$ up to sign).
There are however two other even faces namely (OPQ) and (ZPQ). The equation of the supporting hyperplane is the same in both cases and is $\varphi_2 = \varphi_3$ and one is in case 2) with generic elements of the form $(\varphi_1,\, \varphi_2,\, \varphi_2)$ where for (OPQ) one has $\frac{\pi}{2} > \varphi_1 > \varphi_2 > 0$ while for (ZPQ) one gets $\frac{\pi}{2} + \varphi_2 > \varphi_1 > \frac{\pi}{2} > \varphi_2$. The geometric data determines

$$\sigma(\frac{i+\alpha^{1/2}}{i-\alpha^{1/2}})$$

where $\alpha = \tan\varphi_2 \tan(\varphi_2 - \varphi_1) < 0$. Then $q = \frac{i+\alpha^{1/2}}{i-\alpha^{1/2}}$ is real. This first allows to distinguish these faces from the other even faces treated above. Moreover one checks that

$$\alpha \in]-1,0[\,, \quad \forall \varphi \in (OPQ)\,, \quad \alpha \in]-\infty,-1[\,, \quad \forall \varphi \in (ZPQ)\,.$$

Thus $q > 0$ on (OPQ) and $q < 0$ on (ZPQ) which allows to distinguish these two faces from each other. Finally on each of these faces the geometric data determines α and hence the flow line of φ using lemma 4.8.

Thus we get the proof in all cases except the generic case which we shall now treat in details separately.

6.2. **Basic notations for elliptic curves.**

We recall that given an elliptic curve E viewed as a 1-dimensional complex manifold and chosing a base point $p_0 \in E$ one gets an isomorphism of the universal cover $\tilde{E}$ of E with base point p_0

$$\tilde{E} \overset{I}{\simeq} \mathbb{C}$$

given by the integral

$$I(p) = \int_{p_0}^{p} \omega$$

where ω is a holomorphic $(1,0)$-form. This isomorphism is unique up to multiplication by $\lambda \in \mathbb{C}^*$. Let $L \subset \mathbb{C}$ be the lattice of periods then $I^{-1}(L)$ is the kernel of the covering map $\pi : \tilde{E} \to E$ and one has an isomorphism $E \sim \mathbb{C}/L$. To eliminate the choice of the base point we let $T(E)$ be the group of translations of E and note that the universal cover $\tilde{T}(E)$ identifies with the additive group $\mathbb{C}$,

$$\tilde{T}(E) \simeq \mathbb{C}\,, \quad T(E) \simeq \mathbb{C}/L\,.$$

One can moreover take L of the form $L = \mathbb{Z} + \mathbb{Z}\tau$ where $\tau \in \mathbb{H}/\Gamma[1]$ and $\mathbb{H}$ is the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im} z > 0\}$ while in general $\Gamma[n]$ is the congruence subgroup of level n in $SL(2,\mathbb{Z})$. With $\mathbb{H}^*$ obtained from $\mathbb{H}$ by adjoining the rational points of the boundary one has a canonical isomorphism $\mathbb{H}^*/\Gamma[1] \to \mathbb{P}^1(\mathbb{C})$ given by Jacobi's j function. In terms of the elliptic curve E defined by the equation

$$y^2 = 4x^3 - g_2\, x - g_3\,,$$

in $\mathbb{P}^2(\mathbb{C})$ one has

$$j(E) = 1728\, \frac{g_2^3}{\Delta}$$

where the discriminant is $\Delta = g_2^3 - 27\, g_3^2$.

One obtains a finer invariant $\lambda(E)$ if one has the additional structure given by an isomorphism of abelian groups

$$\phi : \quad (\mathbb{Z}/2\mathbb{Z})^2 \to \frac{1}{2} L/L = T_2(E)\,.$$

where $T_2(E)$ is the group of two torsion elements of $T(E)$. This allows to choose the module τ in a finer manner as $\tau \in \mathbb{H}/\Gamma[2]$ and one has a canonical isomorphism $\mathbb{H}^*/\Gamma[2] \to \mathbb{P}^1(\mathbb{C})$ given by Jacobi's λ function. In terms of the elliptic curve E defined by the equation

$$y^2 = \prod (x - e_j)$$

in $\mathbb{P}^2(\mathbb{C})$ one has

$$\lambda(E) = \mathrm{Cross\,Ratio}(e_1, e_2; e_3, e_4)$$

In more intrinsic terms the labelling of the two torsion $T_2(E)$

$$\omega_j \in T(E)\,,\ \omega_1 = \phi(1,0)\,,\ \omega_3 = \phi(0,1)\,,\ \omega_2 = -\omega_1 - \omega_3\,. \tag{6.5}$$

allows to define the following function on the group $T(E)$ of translations of E:

$$F_\phi(u) = \frac{\wp_3(u)}{\wp_3(\omega_1)}\,, \tag{6.6}$$

where $\wp_3$ is defined using a fixed isomorphism $\tilde{T}(E) \simeq \mathbb{C}$, as the sum

$$\wp_3(u) = \left(\sum_{\pi(y)=u} y^{-2}\right) - \left(\sum_{\pi(y)=\omega_3} y^{-2}\right) \tag{6.7}$$

where one defines the sums by restricting y to $|y| < R$ on both sides and then taking the limit. In standard notation with the Weierstrass $\wp$-function given by

$$\wp(z) = \frac{1}{z^2} + \sum_{\ell \in L^*} \left(\frac{1}{(z+\ell)^2} - \frac{1}{\ell^2}\right) \tag{6.8}$$

one gets

$$\wp_3(u) = \wp(z) - \wp(w_3)\,,\ \pi(z) = u\,. \tag{6.9}$$

Note that the ratios involved in (6.6) eliminate the scale factor λ in the isomorphism $\tilde{T}(E) \simeq \mathbb{C}$. With these notations one has

$$\lambda(E,\phi) = F(w_2) = \frac{\wp(\omega_2) - \wp(\omega_3)}{\wp(\omega_1) - \wp(\omega_3)}\,. \tag{6.10}$$

Finally the covering $\mathbb{H}^*/\Gamma[2] \to \mathbb{H}^*/\Gamma[1]$ is simply given by the algebraic map

$$\lambda \quad \to \quad 256\,\frac{(1-\lambda+\lambda^2)^3}{\lambda^2(1-\lambda)^2}$$

while the group Σ of deck transformations is the dihedral group generated by the two symmetries

$$u(z) = 1/z \quad v(z) = 1 - z\,.$$

One has a unique anti-isomorphism $w:\ PSL(2,\mathbb{Z}/2\mathbb{Z}) \to \Sigma$ such that

$$\lambda(E,\phi\circ\alpha) = w(\alpha)(\lambda(E,\phi))\,. \tag{6.11}$$

Moreover $w(t) = v$ where t is the transposition of $(a,b) \to (b,a)$. Finally one gets,

$$F_{\phi\circ t}(u) = 1 - F_\phi(u)\,. \tag{6.12}$$

6.3. **The generic case.**

We now deal with the generic case.
Let $s_j \in \mathbb{R}$ be three real numbers and (α,β,γ) be given by

$$\alpha = \frac{s_3 - s_2}{s_1}\,,\ \beta = \frac{s_1 - s_3}{s_2}\,,\ \gamma = \frac{s_2 - s_1}{s_3}\,. \tag{6.13}$$

Let (E,σ) be the pair of an elliptic curve and a translation associated to (α,β,γ) by proposition 5.1.

Lemma 6.2. *There exists an isomorphism $\phi:\ (\mathbb{Z}/2\mathbb{Z})^2 \to T_2(E)$ such that with $e_j = s_j^{-1}$ one has*

$$\lambda(E,\phi) = \textit{Cross Ratio}\ (e_2, e_1; e_3, \infty)\,, \tag{6.14}$$

and with F_ϕ defined by (6.6)

$$F_\phi(\sigma) = \frac{s_1}{s_1 - s_3} \tag{6.15}$$

Proof. The proof is straightforward using θ-functions to parametrize the elliptic curve (5.1) but we prefer to give an elementary direct proof. In order to prove (6.14) and (6.15) we start from proposition 5.1 and replace (α, β, γ) by their value. The equations of E simplifies to

$$\sum_0^3 x_\mu^2 = 0\,, \quad \sum_1^3 s_k\, x_k^2 = 0\,. \tag{6.16}$$

We then rescale $x_1 = a\, X_1$, $x_2 = b\, X_2$, $x_3 = X_3$, where

$$a^2 = -\frac{s_3}{s_1}\,, \quad b^2 = -\frac{s_3}{s_2}\,. \tag{6.17}$$

After this rescaling the second equation of (6.16) gives $X_3^2 = X_1^2 + X_2^2$ and one uses the standard rational parametrization of the conic to parametrize the solutions of this equation as

$$x_1 = 2\, a\, t\,,\ x_2 = b(1 - t^2)\,,\ x_3 = (1 + t^2)\,. \tag{6.18}$$

One then writes the first equation in (5.1) as

$$x_0^2 + 4\, a^2\, t^2 + b^2(1 - t^2)^2 + (1 + t^2)^2 = 0 \tag{6.19}$$

and since $1 + b^2 = \dfrac{(s_2 - s_3)}{s_2} \neq 0$ this reduces to the elliptic curve defined by the equation

$$y^2 = t^4 - 2\, r\, t^2 + 1\,, \tag{6.20}$$

where

$$r = -\frac{2\, a^2 - b^2 + 1}{b^2 + 1}\,.$$

Using (6.17) one gets

$$r = \frac{s_1(s_2 + s_3) - 2\, s_2\, s_3}{s_1(s_3 - s_2)}\,. \tag{6.21}$$

One checks that $r \neq \pm 1$ since the s_j are pairwise distinct.
Thus the roots of $x^4 - 2\, r\, x^2 + 1 = 0$ are all distinct and we write them as $\pm\, u$, $\pm\, v$ where

$$uv = 1\,, \quad u^2 + v^2 = 2r\,. \tag{6.22}$$

The cross ratio of $(-v, u; v, -u)$ is independent of the above choices and is given by

$$\frac{-v - v}{-v + u} : \frac{u - v}{u + u} = -\frac{4\, uv}{(u - v)^2} = \frac{-4}{u^2 + v^2 - 2} = -\frac{2}{r - 1} = \frac{1 + b^2}{1 + a^2} = \frac{s_1}{s_2}\, \frac{s_2 - s_3}{s_1 - s_3}\,. \tag{6.23}$$

In other words with $e_j = s_j^{-1}$ one has

$$\text{Cross Ratio}(-v, u; v, -u) = \text{Cross Ratio}(e_2, e_1; e_3, \infty)\,. \tag{6.24}$$

Let then $\gamma \in \mathrm{SL}(2, \mathbb{C})$, $\gamma(X) = \frac{aX + b}{cX + d}$ which transforms (e_2, e_1, e_3, ∞) to $(-v, u, v, -u)$. Then for a suitable choice of $\lambda \neq 0$ the transformation,

$$t = \gamma(X)\,, \quad y = \frac{\lambda Y}{(cX + d)^2}\,.$$

gives a birational isomorphism $\tilde{\gamma}$ of the elliptic curve defined by the equation

$$Y^2 = \prod_1^3 (X - e_i) \tag{6.25}$$

with the curve (6.20). Choosing the origin of (6.25) as the point at infinity one gets the point P with coordinates $t=-u$, $y=0$ as the origin of (6.20). One has $P=(p_0,p_1,p_2,p_3)$ and $p_0=0$ since $y=0$. Thus the other coordinates fulfill (6.16) in the simplified form

$$p_1^2+\,p_2^2+\,p_3^2=\,0\,,\quad s_1\,p_1^2+\,s_2\,p_2^2+\,s_3\,p_3^2=\,0\,, \tag{6.26}$$

and in homogeneous expressions one can replace p_k^2 by $s_\ell-s_m$.

Let us now determine the translation σ. We just compute the t parameter of $\sigma(P)=(x_0,x_1,x_2,x_3)$. The parameter t of (6.18) is recovered in homogeneous coordinates as,

$$t=\frac{b\,x_1}{a\,x_2+a\,b\,x_3}\,. \tag{6.27}$$

One first starts by simplifying (5.9) when applied to P. One has $p_0=0$ and up to an overall scale $(-\alpha\,\beta\,\gamma)$ one gets using (6.26), *i.e.* the replacement $p_k^2\to s_\ell-s_m$,

$$-\beta\,\gamma\,p_1^2+\alpha\,\gamma\,p_2^2+\alpha\,\beta\,p_3^2\to-\,s_1+s_2+s_3$$

$$\beta\,\gamma\,p_1^2-\alpha\,\gamma\,p_2^2+\alpha\,\beta\,p_3^2\to s_1-s_2+s_3$$

$$\beta\,\gamma\,p_1^2+\alpha\,\gamma\,p_2^2-\alpha\,\beta\,p_3^2\to s_1+s_2-s_3$$

Thus the coordinates of $\sigma(P)$ are up to an overall scale

$$x_1=p_1(-s_1+s_2+s_3)\,,\;x_2=p_2(s_1-s_2+s_3)\,,\;x_3=p_3(s_1+s_2-s_3)\,, \tag{6.28}$$

where the p_j are the coordinates of P. These are given by (6.18) taking $t=-u$, with u,v as above, thus,

$$p_1=-2\,a\,u\,,\;p_2=b(1-u^2)\,,\;p_3=1+u^2\,. \tag{6.29}$$

We then need to compute:

$$t(\sigma)=\frac{b\,x_1}{a\,x_2+a\,b\,x_3}=\frac{b(-2\,a\,u)(-s_1+s_2+s_3)}{a\,b(1-u^2)(s_1-s_2+s_3)+a\,b(1+u^2)(s_1+s_2-s_3)}\,. \tag{6.30}$$

We see that a and b drop out and we remain with

$$t(\sigma)=\frac{-\,u(-s_1+s_2+s_3)}{s_1+(s_2-s_3)\,u^2}\,. \tag{6.31}$$

We just need to compute $\mathrm{Cross\,Ratio}(-v,u,v,t(\sigma))$ which is the same as $\mathrm{Cross\,Ratio}(e_2,e_1;e_3,\tau)$ where τ is $\wp(\sigma)$, up to an affine transformation which transforms the e_j to $\wp(\omega_j)$. One has

$$\mathrm{Cross\,Ratio}(-v,u,v,t(\sigma))=\frac{-v-v}{-v-t(\sigma)}:\frac{u-v}{u-t(\sigma)}=\frac{u-t(\sigma)}{u-v}\,\frac{2v}{v+t(\sigma)}=\frac{u^2-u\,t(\sigma)}{u^2-1}\,\frac{2}{1+u\,t(\sigma)}$$

using $u\,v=1$. The $u\,t(\sigma)$ only involves u^2 and one can simplify by its denominator to get

$$\mathrm{Cross\,Ratio}(-v,u,v,t(\sigma))=2\,\frac{(s_2-s_3)\,u^4+(s_2+s_3)\,u^2}{(s_1-2\,s_3)\,u^4+2\,s_3\,u^2-s_1}\,. \tag{6.32}$$

We then claim that one has:

$$\mathrm{Cross\,Ratio}(-v,u,v,t(\sigma))=\frac{s_2-s_3}{s_1-s_3}=\mathrm{Cross\,Ratio}(e_2,e_1;e_3,0)\,. \tag{6.33}$$

i.e. that $\tau = 0$. To see this we replace u^4 by $2\,r\,u^2 - 1$ in the expression (6.32) for the cross ratio, which gives

$$\text{Cross Ratio}(-v, u, v, t(\sigma)) = \frac{(2(s_2 - s_3)\,r + s_2 + s_3)\,u^2 - (s_2 - s_3)}{((s_1 - 2\,s_3)\,r + \,s_3)\,u^2 - (s_1 - s_3)}\,.$$

The computation using (6.21) shows that this fraction is independent of u^2 and equal to $\frac{s_2 - s_3}{s_1 - s_3}$. This ends the proof of the lemma since we have shown that, up to an affine tranformation,

$$\wp(\omega_j) = e_j\,, \quad \wp(\sigma) = \,0\,.$$

so that

$$F_\phi(\sigma) = \,\frac{0 - e_3}{e_1 - e_3} = \,\frac{s_1}{s_1 - s_3}\,.$$

□

We shall now give the proof of theorem 6.1 in the generic case. We start with the alcove A.
Given an elliptic curve E and a translation σ of E we claim that if there is a labelling ϕ of $T_2(E)$ such that

$$\lambda(E, \phi) \in]0, 1[\,, \quad F_\phi(\sigma) < 0 \tag{6.34}$$

then this labelling is unique. Indeed the only element of $PSL(2, \mathbb{Z}/2\mathbb{Z})$ which preserves the first condition is the transposition t and this does not preserve the second by (6.12).
On $\sigma(A) = \{s \,|\, 1 < s_1 < s_2 < s_3\}$ one has (lemma 4.6)

$$\text{Cross Ratio }(s_2, s_1; s_3, 0) \in]0, 1[\,, \quad \frac{s_1}{s_1 - s_3} < 0\,,$$

thus by lemma 6.2 there exists a unique labelling ϕ of $T_2(E)$ such that (6.34) holds. This gives back both Cross Ratio $(s_2, s_1; s_3, 0)$ and $\frac{s_1}{s_1 - s_3}$. The latter gives the ratio $\frac{s_3}{s_1} = a$ and the former then gives $\frac{s_2 - as_1}{s_1 - as_1}\frac{s_1}{s_2} = \frac{s_2 - as_1}{(1 - a)s_2}$ which gives the ratio $\frac{s_2}{s_1} = b$. Thus we recover the flow line using the convexity of $\sigma(A)$.
Let us now look at the alcove B. One has $\sigma(B) = \{s \,|\, s_3 < s_2 < 0, 1 < s_1\}$. Thus $s_2^{-1} < s_3^{-1} < s_1^{-1}$.
Exactly as above, given an elliptic curve E and a translation σ of E we claim that if there is at most one labelling ϕ of $T_2(E)$ such that

$$0 < \,F_\phi(\sigma) \,< \,\lambda(E, \phi) \,< 1\,. \tag{6.35}$$

The algebra associated to the s_j is unchanged, up to isomorphism, by cyclic permutations of the s_j, and the same holds for the associated geometric data. Thus lemma 6.2 gives a labelling ϕ such that

$$\lambda(E, \phi) = \text{Cross Ratio }(s_3, s_2; s_1, 0)\,, \quad F_\phi(\sigma) = \frac{s_2}{s_2 - s_1}\,,$$

One checks that it then fulfills (6.35) since with $e_j = s_j^{-1}$ one has

$$\text{Cross Ratio}(s_3, s_2; s_1, 0) = \,\frac{e_1 - e_3}{e_1 - e_2}\,, \quad \frac{s_2}{s_2 - s_1} = \,\frac{e_1}{e_1 - e_2}\,.$$

and

$$1 > \,\frac{e_1 - e_3}{e_1 - e_2} > \,\frac{e_1}{e_1 - e_2} > 0\,.$$

Thus as above we recover the the ratios $\frac{s_1}{s_2}$ and $\frac{s_3}{s_2}$ and the flow line of φ using the convexity of $\sigma(B)$.
Finally note that the conditions (6.34) and (6.35) are exclusive and thus allow to decide using the geometric data wether $\varphi \in \sigma(A)$ or $\varphi \in \sigma(B)$.

7. Dualities

We show in this section that there are unexpected dualities between the noncommutative spaces $\mathbb{R}^4_\varphi$ in the following cases of Table 5.5

$$A \leftrightarrow B\,, \quad 2 \leftrightarrow 3\,, \quad 5 \leftrightarrow 6\,, \quad 7 \leftrightarrow 8\,, \quad 10 \leftrightarrow 11\,,$$

Modulo these dualities the fundamental domain gets reduced from an alcove of the root system D_3 to the smaller alcove of the root system C_3.

7.1. **Semi-cross product.**

Let $\mathcal{A}$ be a graded algebra and let $\alpha \in \mathrm{Aut}(\mathcal{A})$ be a symmetry commuting with the grading (i.e. homogeneous of degree 0).

Definition 7.1. *The semi-cross product $\mathcal{A}(\alpha)$ of $\mathcal{A}$ by α is the graded vector space $\mathcal{A}$ equipped with the bilinear product $\cdot_\alpha$ defined by*

$$a \cdot_\alpha b = a\,\alpha^n(b), \;\; \forall a \in \mathcal{A}_n\,,\; b \in \mathcal{A}$$

It is easily verified that this product is associative and that $\mathcal{A}_m \cdot_\alpha \mathcal{A}_n \subset \mathcal{A}_{m+n}$ so that $\mathcal{A}(\alpha)$ is a graded algebra and that if $\mathcal{A}$ is unital then $\mathcal{A}(\alpha)$ is also unital with the same unit.
Some basic properties of the semi-cross product are given by the following proposition.

Proposition 7.2. *Let $\mathcal{A}$ be a graded algebra and let α be an automorphism of degree 0 of $\mathcal{A}$.*
(i) If β is an automorphism of degree 0 of $\mathcal{A}$ which commutes with α, then β is also canonically an automorphism of degree 0 of $\mathcal{A}(\alpha)$ and one has

$$\mathcal{A}(\alpha)(\beta) = \mathcal{A}(\alpha \circ \beta).$$

In particular one has

$$\mathcal{A}(\alpha)(\alpha^{-1}) = \mathcal{A}.$$

(ii) If $\mathcal{A}$ is a quadratic algebra, then $\mathcal{A}(\alpha)$ is also a quadratic algebra and its geometric datas $(E', \sigma', \mathcal{L}')$ are deduced from those $(E, \sigma, \mathcal{L})$ of $\mathcal{A}$ as follows

$$E' = E, \;\; \sigma' = \alpha^t \circ \sigma, \;\; \mathcal{L}' = \mathcal{L}$$

where α^t is induced by the transposed of $(\alpha \restriction \mathcal{A}_1)$.
(iii) If $\mathcal{A}$ is an involutive algebra with involution $x \mapsto x^*$ homogeneous of degree 0 (in short, if $\mathcal{A}$ is a graded $*$-algebra) and if α commutes with the involution, then one defines an antilinear antimultiplicative mapping $x \mapsto x^{*_\alpha}$ of $\mathcal{A}(\alpha)$ onto $\mathcal{A}(\alpha^{-1})$ by setting $a^{*_\alpha} = \alpha^{-n}(a^*)$ for $a \in \mathcal{A}_n$. In particular if $\alpha^2 = 1$, then $\mathcal{A}(\alpha)$ equipped with the involution $x \mapsto x^{*_\alpha}$ is a graded $*$-algebra.

Proof. (i) One has for $a \in \mathcal{A}_n$ and $b \in \mathcal{A}$

$$\beta(a \cdot_\alpha b) = \beta(a\alpha^n(b)) = \beta(a)\beta(\alpha^n(b)) = \beta(a)\alpha^n(\beta(b)) = \beta(a) \cdot_\alpha \beta(b)$$

which shows that β is an automorphism of $\mathcal{A}(\alpha)$. One has also

$$a \cdot_\alpha \beta^n(b) = a\alpha^n(\beta^n(b)) = a(\alpha \circ \beta)^n(b) = a \cdot_{\alpha\circ\beta} b$$

which implies $\mathcal{A}(\alpha)(\beta) = \mathcal{A}(\alpha \circ \beta)$.
(ii) Assume that α is a quadratic algebra i.e. $\mathcal{A} = A(V, R) = T(V)/(R)$ where V is finite-dimensional and where (R) is the two-sided ideal of the tensor algebra $T(V)$ of V generated by the subspace R of $V \otimes V$. Let $\mathbf{m}$ denote the product of $\mathcal{A}$ and $\mathbf{m}'$ denote the product of $\mathcal{A}(\alpha)$. Since $V = \mathcal{A}_1 = \mathcal{A}(\alpha)_1$

one has $\mathbf{m}' = \mathbf{m} \circ (\mathrm{Id} \otimes \alpha)$ on $V \otimes V$ and thus $\mathbf{m}(R) = 0$ is equivalent to $\mathbf{m}'((\mathrm{Id} \otimes \alpha^{-1})(R)) = 0$ from which it follows easily that $\mathcal{A}(\alpha) = A(V, R') = T(V)/(R')$ with

$$R' = (\mathrm{Id} \otimes \alpha^{-1})R$$

so $\mathcal{A}(\alpha)$ is quadratic.
By definition the graph of σ' is the subset

$$\Gamma' \subset P(V^*) \times P(V^*)$$

obtained from the subset of $V^* \times V^*$ of pairs (ω, π), $\omega \neq 0, \pi \neq 0$ such that

$$\langle \omega \otimes \pi, r \rangle = 0, \ \ \forall r \in R'.$$

Since $R' = (\mathrm{Id} \otimes \alpha^{-1})R$ we thus get $\sigma' = \alpha^t \circ \sigma$.
(iii) One has for $a \in \mathcal{A}_n$ and $b \in \mathcal{A}_m$

$$\alpha^{-(n+m)}((a \cdot_\alpha b)^*) = \alpha^{-(n+m)}(\alpha^n(b^*)a^*) = \alpha^{-m}(b^*)\alpha^{-m}(\alpha^{-n}(a^*))$$

and thus

$$(a \cdot_\alpha b)^{*_\alpha} = b^{*_\alpha} \cdot_{\alpha^{-1}} a^{*_\alpha}$$

which implies (iii). □

More generally if $\mathcal{A}$ is finitely generated in degree 1 and finitely presented i.e. if $\mathcal{A} = T(V)/(R)$ with V finite-dimensional and (R) is the two-sided ideal of $T(V)$ generated by the graded subspace $R = \oplus_{n=2}^N R_n$ $(R_n \subset V^{\otimes^n})$ of $T(V)$, one has $\mathcal{A}(\alpha) = T(V)/(R(\alpha))$ with $R(\alpha) = \oplus_{n=2}^N R_n(\alpha)$,

$$R_n(\alpha) = (\mathrm{Id} \otimes \alpha^{-1} \otimes \cdots \otimes \alpha^{-(n-1)})R_n.$$

In particular $\mathcal{A}(\alpha)$ is a N-homogeneous algebra whenever $\mathcal{A}$ is N-homogeneous. Let us recall that an algebra $\mathcal{A}$ as above is said to be N-homogeneous iff $R = R_N \subset V^{\otimes^N}$ [4], [5]. For these algebras, which generalize the quadratic algebras ($N = 2$), one has a direct extension of the Koszul duality of quadratic algebras as well as a natural generalization of the notion of Koszulity [4], [5]. The stability of the corresponding homological notions with respect to the semi-cross product construction will be studied elsewhere.

The terminology semi-cross product of $\mathcal{A}$ by α for $\mathcal{A}(\alpha)$ comes from the fact that it can be identified with a subalgebra of the crossed product $\mathcal{A} \rtimes_\alpha \mathbb{Z}$, namely the subalgebra generated by the elements

$$xW, \ \ x \in \mathcal{A}_1$$

where W denotes the new invertible generator of the crossed product defined by $WaW^{-1} = \alpha(a)$ for $a \in \mathcal{A}$. Indeed one has for $a \in \mathcal{A}_m$ $b \in \mathcal{A}_p$

$$aW^n bW^p = a\alpha^n(b)\, W^{n+p}$$

If $\mathcal{A}$ is a graded $*$-algebra with α a $*$-homomorphism of degree 0, one endows the crossed product of a structure of $*$-algebra by setting $W^* = W^{-1}$. The involution of the crossed product induces then, by restriction, the antilinear antimultiplicative mapping $*_\alpha : \mathcal{A}(\alpha) \to \mathcal{A}(\alpha^{-1})$ of (iii) in the above proposition.
For the following application, we shall have $\alpha^2 = 1$ so $\mathcal{A}(\alpha)$ can then be identified with the corresponding subalgebra of the crossed product $\mathcal{A} \rtimes_\alpha \mathbb{Z}/2\mathbb{Z}$ and if $\mathcal{A}$ is a graded $*$-algebra with α a $*$-homomorphism of degree 0, $\mathcal{A} \rtimes_\alpha \mathbb{Z}/2\mathbb{Z}$ becomes a $*$-algebra by setting $W = W^*(= W^{-1})$ and $\mathcal{A}(\alpha)$ is a $*$-subalgebra.

7.2. Application to $\mathbb{R}^4_\varphi$ and S^3_φ.

The above construction allows to give a duality between the following cases of Table 5.5

$$A \leftrightarrow B\,,\quad 2 \leftrightarrow 3\,,\quad 5 \leftrightarrow 6\,,\quad 7 \leftrightarrow 8\,,\quad 10 \leftrightarrow 11\,,$$

where A and B are the two simplices that together complete case 1).
The explicit transformation on the φ-parameters is

$$f_1(\varphi) = (\pi - \varphi_1,\, \frac{\pi}{2} - \varphi_1 + \varphi_2,\, \frac{\pi}{2} - \varphi_1 + \varphi_3)\,. \tag{7.1}$$

but there are two others which give relevant additional dualities,

$$f_2(\varphi) = (\frac{\pi}{2} + \varphi_2 - \varphi_3,\, \varphi_2,\, \frac{\pi}{2} - \varphi_1 + \varphi_2)\,. \tag{7.2}$$

$$f_3(\varphi) = (\frac{\pi}{2} - \varphi_2 + \varphi_3,\, \frac{\pi}{2} - \varphi_1 + \varphi_3,\, \varphi_3)\,. \tag{7.3}$$

They are all involutions $f_j^2 = 1$ and on the fundamental domain $A \cup B$ one has

$$f_1(A) = B\,,\quad f_1(B) = A\,,\quad f_2(B) = B\,,\quad f_3(A) = A\,.$$

with $f_1(0PQP'Z) = (ZPQP'0)$, $f_2(PQP'Z) = (PQZP')$ and $f_3(OPQP') = (P'PQO)$. Thus the duality f_2 (resp. f_3) operates only in B (resp. A). Moreover the transformations f_2 and f_3 are conjugate under f_1 *i.e.* $f_2 = f_1 \circ f_3 \circ f_1$.

By proposition 2.4 2) any element v of the centralizer $C \subset \mathrm{SO}(4)$ of the diagonal matrices in SU(4) defines an automorphism of $\mathbb{R}^4_\varphi$ acting by v on the generators. We thus let $\alpha_j \in \mathrm{Aut}\mathcal{A}$ correspond to the diagonal matrices with respective diagonals given by

$$\alpha_1 \,:\, (1,1,-1,-1)\,,\quad \alpha_2 \,:\, (1,-1,1,-1)\,,\quad \alpha_3 \,:\, (1,-1,-1,1)\,, \tag{7.4}$$

Proposition 7.3. *For $j \in \{1,2,3\}$ one has canonical $*$-isomorphisms*

$$\rho_j : C_{\mathrm{alg}}(\mathbb{R}^4_{f_j(\varphi)}) \to C_{\mathrm{alg}}(\mathbb{R}^4_\varphi)(\alpha_j)$$

which preserve the central element $\sum x_\mu^2$ and induce corresponding isomorphisms $S^3_{f_j(\varphi)} \simeq S^3_\varphi(\alpha_j)$.

Proof. One writes explicitely the isomorphisms as follows on the canonical (self-adjoint) generators x_j^μ of $C_{\mathrm{alg}}(\mathbb{R}^4_{f_j(\varphi)})$, (which are the canonical noncommutative coordinates of $\mathbb{R}^4_{f_j(\varphi)}$)

$$\rho_1(x_1^0) = x^1\,W_1\,,\quad \rho_1(x_1^1) = x^0\,W_1\,,\quad \rho_1(x_1^2) = -i\,x^2\,W_1\,,\quad \rho_1(x_1^3) = -i\,x^3\,W_1\,,$$

$$\rho_2(x_2^0) = x^2\,W_2\,,\quad \rho_2(x_2^1) = -i\,x^3\,W_2\,,\quad \rho_2(x_2^2) = x^0\,W_2\,,\quad \rho_2(x_2^3) = i\,x^1\,W_2\,,$$

$$\rho_3(x_3^0) = x^3\,W_3\,,\quad \rho_3(x_3^1) = -i\,x^2\,W_3\,,\quad \rho_3(x_3^2) = i\,x^1\,W_3\,,\quad \rho_3(x_3^3) = x^0\,W_3\,,$$

Using the signs in (7.4) one checks that one gets $*$-isomorphisms $C_{\mathrm{alg}}(\mathbb{R}^4_{f_j(\varphi)}) \simeq C_{\mathrm{alg}}(\mathbb{R}^4_\varphi)(\alpha_j)$ and that one has $\rho_j(\sum_\mu (x_j^\mu)^2) = \sum_\mu (x^\mu)^2$. ∎

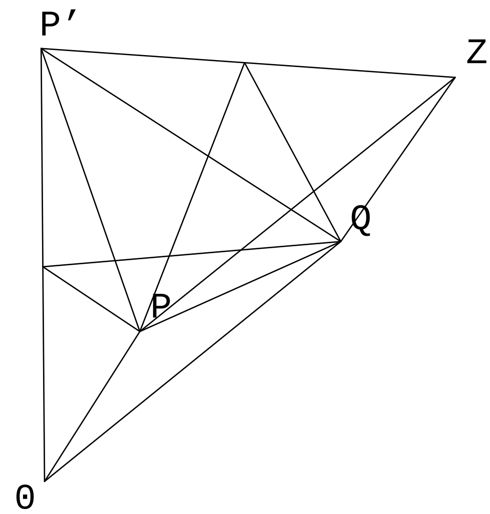

FIGURE 3. The hyperplanes $f_j(\varphi) = \varphi$

Corollary 7.4. (1) *Let φ be generic and $(E, \sigma, \mathcal{L})$ be the geometric data of $C_{\mathrm{alg}}(\mathbb{R}^4_\varphi)$. Then the geometric data of $C_{\mathrm{alg}}(\mathbb{R}^4_{f_j(\varphi)})$ is $(E, \sigma + \tau_j, \mathcal{L})$ where $\tau_j \in T_2(E)$.*
(2) *The hyperplane $f_j(\varphi) = \varphi$ is globally invariant under the flow Z.*
(3) *Let $\varphi \in A \cup B$ be generic, then it belongs to the hyperplane $f_j(\varphi) = \varphi$ (with $j = 2$ on B and $j = 3$ on A) iff the translation σ fulfills*

$$\sigma \in T_4(E)\,.$$

Proof. 1) Using proposition 7.2 the required equality follows if one shows that the action of α_j on E is indeed given by a translation $\tau_j \in T_2(E)$. This can be checked directly using ϑ-functions *i.e.* proposition 9.3.

□

These new symmetries suggest that one extends the group $\Gamma(\mathbf{T}) \rtimes W$ of section 3.3 to include the T_j. This amounts to adding the following new roots to the root system Δ of section 3.3. To the $\pm\psi_\mu \pm \psi_\nu$ we adjoin the $\pm 2\psi_\mu$. This is in fact the same as the replacement

$$SO(6) \to Sp(3)\,,$$

of the compact group $SO(6)$ by the symplectic group $Sp(3)$. (*i.e.* of D_3 by C_3). The corresponding Weyl group is now $O(3, \mathbb{Z})$.

The description of $Sp(3)$ is obtained as follows using the natural representation $\rho : \mathbb{H} \to M_2(\mathbb{C})$ of the field of quaternions $\mathbb{H}$ as two by two matrices of the form $\begin{bmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{bmatrix}$ with $\alpha, \beta \in \mathbb{C}$. By definition $Sp(3)$ is the group of three by three matrices $Q \in M_3(\mathbb{H})$ whose image $\rho(Q) \in M_6(\mathbb{C})$ is unitary. Thus it contains as a subgroup

$$\{Q \in M_3(\mathbb{H}) \,|\, \rho(Q) \in U(6) \cap M_6(\mathbb{R})\}$$

which is isomorphic to $U(3)$ and is contained in $SO(6)$ but has only 9 parameters.

8. The algebras $C_{\mathrm{alg}}(\mathbb{R}^4_\varphi)$ in the nongeneric cases

In this section we shall describe the algebras $C_{\mathrm{alg}}(\mathbb{R}^4_\varphi)$ in the nongeneric cases using the above dualities.

8.1. $U_q(\mathfrak{sl}(2))$, $U_q(\mathfrak{su}(2))$ and their homogenized versions.

In this section for $q \in \mathbb{C}_*$, $U_q(\mathfrak{sl}(2))$ is considered as an associative algebra while for $|q| = 1$ or $q \in \mathbb{R}_*$, $U_q(\mathfrak{su}(2))$ is considered as a $*$-algebra which is a real form of $U_q(\mathfrak{sl}(2))$. The coalgebra aspect plays no role in the following and we refer to [21] for a very complete discussion of these topics. It is convenient to start by the homogenized version. For $q \in \mathbb{C}\backslash\{-1, 0, 1\}$ we define $U_q(\mathfrak{sl}(2))^{\mathrm{hom}}$ to be the quadratic algebra generated by 4 elements X^+, X^-, K^+, K^- with relations

$$K^+K^- = K^-K^+ \tag{8.1}$$

$$K^+X^+ = qX^+K^+ \tag{8.2}$$

$$K^+X^- = q^{-1}X^-K^+ \tag{8.3}$$

$$K^-X^+ = q^{-1}X^+K^- \tag{8.4}$$

$$K^-X^- = qX^-K^- \tag{8.5}$$

$$[X^+, X^-] = \frac{(K^+)^2 - (K^-)^2}{q - q^{-1}} \tag{8.6}$$

The "classical limit" $U(\mathfrak{sl}(2))^{\mathrm{hom}} = U_1(\mathfrak{sl}(2))^{\mathrm{hom}}$ for $q = 1$ is obtained by setting $q = 1 + \varepsilon$ and $K^\pm = X^0 \pm \frac{\varepsilon}{2}X^3$. Letting $\varepsilon \to 0$, the relations read then

$$[X^0, X^3] = 0 \tag{8.7}$$

$$[X^0, X^+] = 0 \tag{8.8}$$

$$[X^0, X^-] = 0 \tag{8.9}$$

$$[X^3, X^+] = 2X^0X^+ \tag{8.10}$$

$$[X^3, X^-] = -2X^0X^- \tag{8.11}$$

$$[X^+, X^-] = X^0X^3 \tag{8.12}$$

To obtain $U_q(\mathfrak{sl}(2))$, one notices that relations (8.1) ... (8.5) imply that K^+K^- is central so that one may add the inhomogeneous relation

$$K^+K^- = \mathbb{1} \tag{8.13}$$

which together with relations (8.1) ... (8.6) defines $U_q(\mathfrak{sl}(2))$, i.e. $U_q(\mathfrak{sl}(2))$ is (as associative algebra) the quotient of $U_q(\mathfrak{sl}(2))^{\text{hom}}$ by the two-sided ideal generated by $K^+K^- - \mathbb{1}$.
Similarily one notices that relations (8.7) ... (8.11) imply that X^0 is central and the universal enveloping algebra $U(\mathfrak{sl}(2))$ is obtained by adding $X^0 = 1$ to the relations (8.7) ... (8.12). One has of course $\lim_{q\to 1} U_q(\mathfrak{sl}(2)) = U(\mathfrak{sl}(2))$.
For $q \in \mathbb{C}\backslash\mathbb{R}$ with $|q| = 1$ the real version $U_q(\mathfrak{su}(2))^{\text{hom}}$ of $U_q(\mathfrak{sl}(2))^{\text{hom}}$ is obtained by endowing the algebra with the unique antilinear antimultiplicative involution such that

$$(X^\pm)^* = X^\mp \tag{8.14}$$

$$(K^\pm)^* = K^\mp \tag{8.15}$$

while for $q \in \mathbb{R}\backslash\{-1, 0, 1\}$, $U_q(\mathfrak{su}(2))^{\text{hom}}$ corresonds to the unique antilinear antimultiplicative involution such that

$$(X^\pm)^* = X^\mp$$

$$(K^\pm)^* = K^\pm \tag{8.16}$$

which gives

$$(X^\pm)^* = X^\mp$$

$$(X^0)^* = X^0 \tag{8.17}$$

$$(X^3)^* = X^3 \tag{8.18}$$

for the limiting case $q = 1$, i.e. for $U(\mathfrak{su}(2))^{\text{hom}}$.
These involutions pass to the quotient to define the $*$-algebras $U_q(\mathfrak{su}(2))$ for $|q| = 1$ or $q \in \mathbb{R}_*$. Notice that the involution of $U_q(\mathfrak{su}(2))$ is obtained from the one of $U(\mathfrak{su}(2))$ by setting $K^\pm = q^{\pm\frac{1}{2}X^3}$ (the relations in terms of $X^\pm$ and X^3 differ of course).

8.2. **The algebras in the nongeneric cases.**

We now identify the algebra $C_{\text{alg}}(\mathbb{R}^4_\varphi)$ in the cases 2 to 11 of the table 5.5.

2. **Even face** : $\varphi_1 = \varphi_2 = \varphi$, $\varphi - \varphi_3 \notin \frac{\pi}{2}\mathbb{Z}$, $\varphi_k \notin \frac{\pi}{2}\mathbb{Z}$. By setting

$$X^\pm = x^1 \pm ix^2 \tag{8.19}$$

the relations (2.24), (2.25) of the algebra read then

$$[x^0, x^3] = 0 \tag{8.20}$$

$$\cos(\varphi)[x^0, X^+] = \sin(\varphi - \varphi_3)(x^3X^+ + X^+x^3) \tag{8.21}$$

$$\cos(\varphi-\varphi_3)[x^3,X^+]=-\sin(\varphi)(x^0X^++X^+x^3) \tag{8.22}$$

$$\cos(\varphi)[x^0,X^-]=-\sin(\varphi-\varphi_3)(x^3X^-+X^-x^3) \tag{8.23}$$

$$\cos(\varphi-\varphi_3)[x^3,X^-]=\sin(\varphi)(x^0X^-+X^-x^0) \tag{8.24}$$

$$[X^+,X^-]=-2\sin(\varphi_3)(x^0x^3+x^3x^0) \tag{8.25}$$

and since (8.19) implies $(X^+)^*=X^-$ one sees that relations (8.21) and (8.22) are the adjoints of relations (8.23) and (8.24) respectively. We now distinguish the following 2 regions R and R'.

$R:\frac{\pi}{2}>\varphi>\varphi_3\geq 0$. By setting

$$K^\pm=(2\tan(\varphi_3))^{1/2}((\sin(2\varphi))^{1/2}x^0\pm i(\sin(2(\varphi-\varphi_3)))^{1/2}x^3) \tag{8.26}$$

the relations (8.20) to (8.25) become (8.1) to (8.6) with

$$q=\frac{1-i(\tan(\varphi-\varphi_3)\tan(\varphi))^{1/2}}{1+i(\tan(\varphi-\varphi_3)\tan(\varphi))^{1/2}} \tag{8.27}$$

so $C_{\mathrm{alg}}(\mathbb{R}^4_\varphi)$ coincides then with $U_q(\mathfrak{su}(2))^{\mathrm{hom}}$ for $|q|=1$, $q\neq\pm 1$ as $*$-algebra (one has $(K^+)^*=K^-$).

$R':\frac{\pi}{2}>\varphi>0$ and $\frac{\pi}{2}+\varphi>\varphi_3>\varphi$. By setting

$$K^\pm=(2\tan(\varphi_3))^{1/2}((\sin(2\varphi))^{1/2}x^0\pm(\sin(2(\varphi_3-\varphi)))^{1/2}x^3) \tag{8.28}$$

the relations (8.20) to (8.25) become (8.1) to (8.6) with

$$q=\frac{1-(\tan(\varphi_3-\varphi)\tan(\varphi))^{1/2}}{1+(\tan(\varphi_3-\varphi)\tan(\varphi))^{1/2}} \tag{8.29}$$

so $C_{\mathrm{alg}}(\mathbb{R}^4_\varphi)$ coincides then with $U_q(\mathfrak{su}(2))^{\mathrm{hom}}$ for $q\in]-1,0[\cup]0,1[$ (one has $(K^\pm)^*=K^\pm$ here).

In general for case 2, it is easy to see that $C_{\mathrm{alg}}(\mathbb{R}^4_\varphi)$ is isomorphic as $*$-algebra either to an algebra of R or of R'. Notice that $q>1$, for instance, is the same as $q\in]0,1[$ by echanging q and q^{-1} and K^+ and K^-. Thus for case 2 the $C_{\mathrm{alg}}(\mathbb{R}^4_\varphi)$ are the $U_q(\mathfrak{su}(2))^{\mathrm{hom}}$.

3. **Odd face** : $\varphi_1=\frac{\pi}{2}$, $\varphi_2-\varphi_3\notin\frac{\pi}{2}\mathbb{Z}$, $\varphi_2\notin\frac{\pi}{2}\mathbb{Z}$, $\varphi_3\notin\frac{\pi}{2}\mathbb{Z}$. Using the analysis of Section 7, one sees that the cases corresponding to the plane $\varphi_1=\frac{\pi}{2}$ are in α_3-duality with the cases corresponding to the plane $\varphi_2=\varphi_3$. On the other hand, the cases corresponding to the plane $\varphi_2=\varphi_3$ are the same as the cases corresponding to the plane $\varphi_1=\varphi_2$. So finally (taking into account the forbidden values $\frac{\pi}{2}\mathbb{Z}$) one sees that the case 3 (odd face) is obtained by duality (in the sense of Section 9) from the case 2 (even face) i.e. from the $U_q(\mathfrak{su}(2))^{\mathrm{hom}}$.

4. $\alpha\perp\beta$ $(0,1)$: $\varphi_1=\frac{\pi}{2}$, $\varphi_2=\varphi_3=\varphi\notin\frac{\pi}{2}\mathbb{Z}$. This case is singular in the sense that one of the 6 relations is missing namely

$$\cos(\varphi_1)[x^0,x^1]=i\sin(\varphi_2-\varphi_3)[x^2,x^3]_+$$

which gives “0=0”. This implies exponential growth. This is a singular limiting case of $U_q(\mathfrak{su}(2))^{\mathrm{hom}}$ for $q = 0$ which separates the regions $1 < q < 0$ and $0 < q < 1$ of case 2.

5. $\alpha - \beta$ $(0,1)$: $\varphi_1 = \frac{\pi}{2}$, $\varphi_2 = \frac{\pi}{2}$, $\varphi_3 \notin \frac{\pi}{2}\mathbb{Z}$. This case is obtained by α_3-duality (section 7) from $\varphi_1 = \varphi_2 = \varphi_3 \notin \frac{\pi}{2}\mathbb{Z}$ which is case 6 below. It corresponds to $U_{-1}(\mathfrak{su}(2))^{\mathrm{hom}}$.

6. $\alpha - \beta$ $(0,0)$: $\varphi_1 = \varphi_2 = \varphi_3 \notin \frac{\pi}{2}\mathbb{Z}$. The relation (2.24), (2.25) reduce in this case to the relations (8.7) to (8.12) by setting $X^{\pm} = x^1 \pm ix^2$, $X^3 = 2x^3$ and $X^0 = -2\sin(\varphi_1)x^0$. Thus in this case the $*$-algebra is isomorphic to $U(\mathfrak{su}(2))^{\mathrm{hom}} = U_1(\mathfrak{su}(2))^{\mathrm{hom}}$.

7. $\alpha \perp \beta$ $(0,0)$: $\varphi_1 = \varphi_2 = -\frac{1}{2}\theta \notin \frac{\pi}{2}\mathbb{Z}$, $\varphi_3 = 0$. This is the “θ-deformation” studied in [17] and [13]. By setting

$$z^1 = x^0 + ix^3,\ z^2 = x^1 + ix^2,\ \bar{z}^1 = (z^1)^*,\ \bar{z}^2 = (z^2)^* \tag{8.30}$$

the relations (2.24), (2.25) read

$$\begin{cases} z^1z^2 = e^{i\theta}z^2z^1 \\ \bar{z}^1z^2 = e^{-i\theta}z^2\bar{z}^1 \\ z^1\bar{z}^2 = e^{-i\theta}\bar{z}^2z^1 \\ \bar{z}^1\bar{z}^2 = e^{i\theta}\bar{z}^2\bar{z}^1 \end{cases} \tag{8.31}$$

and we refer to Part I [13] for more details and generalizations of this algebra.

8. $\alpha \perp \beta$ $(1,1)$: $\varphi_1 = \frac{\pi}{2} + \varphi$, $\varphi_2 = \frac{\pi}{2}$, $\varphi_3 = \varphi \notin \frac{\pi}{2}\mathbb{Z}$. This case is obtained from the preceding case 7 (θ-deformation) by α_1-duality as explained in section 7.

9. $\varphi_1 = \varphi_2 = \varphi_3 = \frac{\pi}{2}$. This is the most singular case, 3 relations are missing (i.e. reduce to “0=0”) among the 6 relations (2.24), (2.25). The algebra has exponential growth.

10. $\varphi_1 = \varphi_2 = \frac{\pi}{2}$, $\varphi_3 = 0$. This case which is at the intersection of the lines carrying case 7 and case 8 is obtained by α_3-duality (section 7) from the next case 11.

11. $\varphi_1 = \varphi_2 = \varphi_3 = 0$. This is the “classical case”, the relations (2.24), (2.25) reduce to $x^\mu x^\nu = x^\nu x^\mu$ for $\mu, \nu \in \{0,1,2,3\}$ so the algebra reduces to the algebra of polynomials $\mathbb{C}[x^0, x^1, x^2, x^3]$.

Remark : One can go much further and describe the C^*-algebras corresponding to the noncommutative spheres S^3_φ in all these degenerate cases. It is important in that respect to classify the discrete series besides the obvious continuous series of representations.

9. The Complex Moduli Space and its Net of Elliptic Curves

We first explain in this section a striking coincidence in the generic case between the geometric data of $\mathbb{R}^4_\varphi$ and the fiber of the double cover 4.6. This takes place in the real moduli space and leads us to introduce the complex moduli space in which the equality between the two elliptic curves makes full sense. We then describe the geometric structure of the complex moduli space as a net of elliptic

curves in three dimensional projective space and exhibit a presentation of the algebras making the equality "fiber = characteristic" manifest.
We showed above in the proof of lemma 6.2 that the geometric data of $\mathbb{R}^4_\varphi$ can be interpreted as the elliptic curve

$$E_2 = \{(X,Y)\,|\,Y^2 = \prod_1^3 (X - e_i)\} \tag{9.1}$$

with $e_j = s_j^{-1}$ and endowed with a translation σ sending the point at ∞ to a point of E_2 whose X coordinate vanishes[5].
One can write E_2 in the equivalent form,

$$E_3(\varphi) = \{(X,Y)\,|\,Y^2 = \prod_1^3 (X\,s_j - 1)\} \tag{9.2}$$

In this form this equation is the same as the one involved in the double cover 4.6. More precisely one gets

Proposition 9.1. *Let φ be generic, and*

$$\text{Fiber}\,(\varphi) = \{\varphi' \,\text{generic}\,|\, J_{\ell m}(\varphi') = J_{\ell m}(\varphi)\,,\ \forall k\}$$

a) There is a natural isomorphism ℓ_φ : $\text{Fiber}\,(\varphi) \simeq E_3(\varphi) \cap (\mathbb{R}\times\mathbb{R}^*)$ determined by the equalities

$$X(\varphi') = s_j(\varphi')/s_j\,,\ \forall j\,, \quad Y(\varphi') = \prod \tan(\varphi'_j)\,. \tag{9.3}$$

b) The closure $\overline{\text{Fiber}}\,(\varphi)$ is the union $\text{Fiber}\,(\varphi) \cup W(P)$ and ℓ_φ extends to an isomorphism

$$\ell_\varphi\ :\ \overline{\text{Fiber}}\,(\varphi) \to E_3(\varphi) \cap \mathbb{P}_2(\mathbb{R})\,.$$

Proof. a) Let $\varphi' \in \text{Fiber}\,(\varphi)$ then by lemma 4.7 the ratio $X = s_j(\varphi')/s_j$ is independent of j. Moreover one has

$$\prod_1^3 (X\,s_j - 1) = \Big(\prod \tan(\varphi'_j)\Big)^2\,,$$

thus the map is well defined. Conversely given $(X,Y) \in E_3(\varphi) \cap (\mathbb{R}\times\mathbb{R}^*)$ one lets $s'_k = X\,s_k$ and $t'_k = Y(s'_k - 1)^{-1}$. This determines uniquely the φ'_j such that $\tan(\varphi'_j) = t'_j$.
b) Note that $\text{Fiber}\,(\varphi)$ is not connected but falls in four connected components each a flow line of the scaling flow X. These correspond to the four components of $E_3(\varphi) \cap (\mathbb{R}\times\mathbb{R}^*)$ *i.e.* of the set of real points of $E_3(\varphi)$ which are not of two torsion. The inverse map ℓ_φ^{-1} extends to the four points $\{\infty, e_1, e_2, e_3\}$ of $E_3(\varphi)$ and one thus obtains a compactification of $\text{Fiber}\,(\varphi)$ by adding the four points $\{P,Q,R,S\}$ in the orbit $W(P)$. The above map ℓ_φ extends continuously to the compactification and the four points $\{P,Q,R,S\}$ map to the four two torsion points $\{\infty, e_1, e_2, e_3\}$ of $E_3(\varphi)$ and in that order for $\varphi \in A$ as in Figure 4. The situation is similar on B but the order of e_2 and e_3 is reversed as well as that of R and S. Moreover the point 0 is now between Q and S instead of being on the left of S as in case $\varphi \in A$ (see Figure 4). ∎

In fact the action of the discrete symmetry given by the Klein group $H \subset W$ preserves globally $\text{Fiber}\,(\varphi)$ since it does not alter the $J_{\ell m}(\varphi)$. In fact this discrete symmetry admits a simple interpretation in terms of translations of order two $\tau \in T_2(E_3)$ as follows, where we let $k_j \in H$ be given

[5] the two choices give isomorphic pairs (E_2, σ)

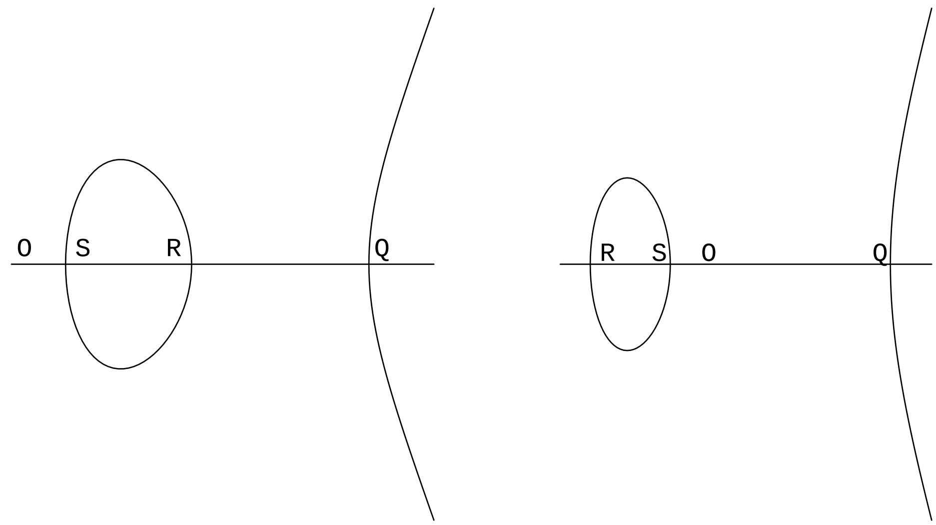

FIGURE 4. The elliptic curve E_3 and the fiber for $\varphi \in A$ and $\varphi \in B$.

by

$$k_1(\varphi) = (-\varphi_1, \varphi_3 - \varphi_1, \varphi_2 - \varphi_1)$$

Proposition 9.2. 1) *Given* $\varphi' \in \mathrm{Fiber}(\varphi)$ *the map* $(X, Y) \to (\lambda X, Y)$ *for* $\lambda = s_j(\varphi')/s_j$, $\forall j$ *is an isomorphism* $E_3(\varphi') \simeq E_3(\varphi)$ *compatible with the isomorphisms* ℓ_φ *and* $\ell_{\varphi'}$ *of lemma 9.1.*

2) $\mathrm{Fiber}(\varphi)$ *is globally invariant under* k_j *and under the above isomorphism* $\mathrm{Fiber}(\varphi) \simeq \mathbb{R}^2 \cap E_3(\varphi)$ *the action of* k_j *is given by the translation by the two torsion element* $(e_j, 0) \in E_3(\varphi)$.

Proof. 1) By construction the $s_j(\varphi')$ are proportional to the s_j so the conclusion follows looking at the definition 9.3 of the identification $\mathrm{Fiber}(\varphi) \simeq \mathbb{R}^2 \cap E_3(\varphi)$.
2) Using 1) it is enough to show that the point of $E_3(\varphi)$ associated to $k_1(\varphi)$ is obtained from $(1, t_1\, t_2\, t_3) \in E_3(\varphi)$ (with $t_j = \tan(\varphi_j)$) by the translation of order two associated to $(e_1, 0) \in E_3(\varphi)$. One checks that the effect of k_1 on the $s_j(\varphi)$ is to multiply all of them by $X' = (1 + t_1^2)/(s_2\, s_3)$. Its effect on $Y = \prod \tan(\varphi_j)$ is to replace it by

$$Y' = -t_1\,(t_3 - t_1)\,(t_2 - t_1)/(s_2\, s_3)\,.$$

One then checks by direct computation that the line joining the points $(1, t_1\, t_2\, t_3) \in E_3(\varphi)$ and $(X', -Y') \in E_3(\varphi)$ intersects $E_3(\varphi)$ in the other point $(e_1, 0)$. The result follows since colinearity of three points A, B, C on the elliptic curve E_3 means $A + B + C = 0$ in that abelian group, while the opposite of $A = (X, Y)$ is $-A = (X, -Y)$. $\square$

The above proposition shows that one should give a direct definition of $E_3(\varphi)$ depending only upon the $J_{\ell m}(\varphi)$ rather than the $s_j(\varphi)$. There is also a strong reason to define in a natural manner the complex points of $E_3(\varphi)$. Indeed the translation σ corresponds to points where the coordinate $X = 0$ and the corresponding equation for Y is $Y^2 = -1$ which does not admit real solutions.

9.1. **Complex moduli space $\mathcal{M}_{\mathbb{C}}$.**

We shall show now that there is a natural way of extending the moduli space from the real to the complex domain. The $E_3(\varphi)$ then appear as a net of degree 4 elliptic curves in $P_3(\mathbb{C})$ having 8 points

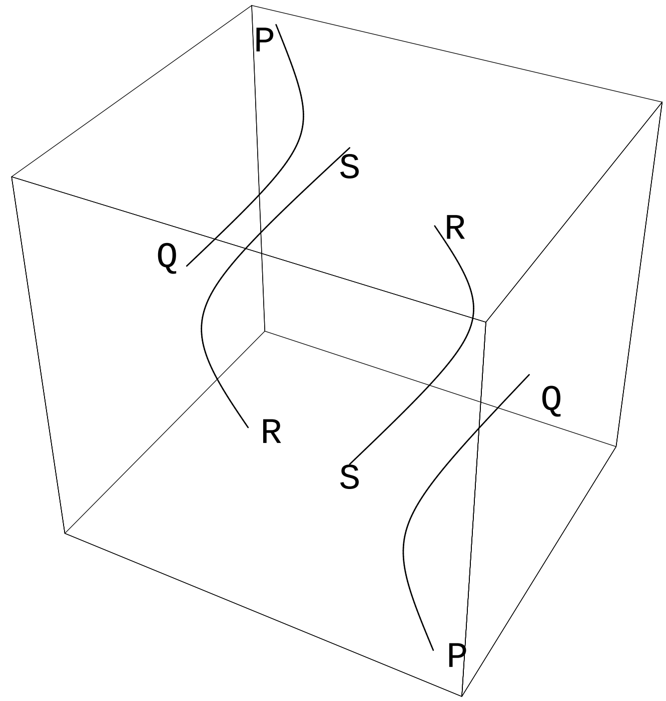

FIGURE 5. The flow lines

in common. These elliptic curves will turn out to play a fundamental role and to be closely related to the elliptic curves of the geometric data of the quadratic algebras which their elements label.

To extend the moduli space to the complex domain we start with the relations defining the involutive algebra $C_{\rm alg}(S^3(\Lambda))$ and take for Λ the diagonal matrix with

$$\Lambda^\mu_\mu := u_\mu^{-1} \tag{9.4}$$

where (u_0, u_1, u_2, u_3) are the coordinates of $\mathbf{u} \in P_3(\mathbb{C})$. Using $y_\mu := \Lambda^\mu_\nu z^\nu$ one obtains the homogeneous defining relations in the form,

$$\begin{aligned} u_k\, y_k\, y_0 - u_0\, y_0\, y_k + u_\ell\, y_\ell\, y_m - u_m\, y_m\, y_\ell &= 0 \\ u_k\, y_0\, y_k - u_0\, y_k\, y_0 + u_m\, y_\ell\, y_m - u_\ell\, y_m\, y_\ell &= 0 \end{aligned} \tag{9.5}$$

for any cyclic permutation (k, ℓ, m) of (1,2,3). The inhomogeneous relation becomes,

$$\sum u_\mu\, y_\mu^2 = 1 \tag{9.6}$$

and the corresponding algebra $C_{\rm alg}(S^3_{\mathbb{C}}(\mathbf{u}))$ only depends upon the class of $\mathbf{u} \in P_3(\mathbb{C})$ (see the remark at the end of 9.2). We let $C_{\rm alg}(\mathbb{C}^4(\mathbf{u}))$ be the quadratic algebra defined by the six relations (9.5). Taking $u_\mu = e^{2i\,\varphi_\mu}$, $\varphi_0 = 0$, for all μ and $x^\mu := e^{i\,\varphi_\mu} y_\mu$ we obtain the defining relations of $C_{\rm alg}(\mathbb{R}^4_\varphi)$ (except for $x^{\mu *} = x^\mu$ which allows to pass from $\mathbb{C}^4(\mathbf{u})$ to $\mathbb{R}^4_\varphi$). Thus the torus $\mathbb{T}_A$ sits naturally in $P_3(\mathbb{C})$ as

$$\mathbb{T}_A = \{\mathbf{u} \in P_3(\mathbb{C}) \mid |u_\mu| = |u_\nu|\,, \quad \forall \mu,\, \nu\} \tag{9.7}$$

In terms of homogeneous parameters the functions $J_{\ell m}(\varphi)$ read as

$$J_{\ell m}(\varphi) = \tan(\varphi_0 - \varphi_k)\tan(\varphi_\ell - \varphi_m) \tag{9.8}$$

for any cyclic permutation (k,ℓ,m) of (1,2,3), and extend to the complex domain $\mathbf{u}\in P_3(\mathbb{C})$ as,

$$J_{\ell m}(\mathbf{u})=\frac{(u_0+u_\ell)(u_m+u_k)-(u_0+u_m)(u_k+u_\ell)}{(u_0+u_k)(u_\ell+u_m)} \tag{9.9}$$

It follows easily from the argument of proposition 4.5 that for generic values of $\mathbf{u}\in P_3(\mathbb{C})$ the quadratic algebra $C_{\rm alg}(\mathbb{C}^4(\mathbf{u}))$ only depends upon $J_{k\ell}(\mathbf{u})$. We thus define the *complex* fiber as

$$F(\mathbf{u}):=\{\mathbf{v}\in P_3(\mathbb{C})\,|\,J_{k\ell}(\mathbf{v})=\ J_{k\ell}(\mathbf{u})\} \tag{9.10}$$

Let then,

$$\Phi(\mathbf{u})=(a,b,c)=\{(u_0+u_1)(u_2+u_3),(u_0+u_2)(u_3+u_1),(u_0+u_3)(u_1+u_2)\} \tag{9.11}$$

be the three roots of the Lagrange resolvent of the 4th degree equation $\prod(x-u_j)=0$. We view Φ as a map

$$\Phi: P_3(\mathbb{C})\backslash S\to P_2(\mathbb{C}) \tag{9.12}$$

where S is the following set of 8 points

$$\begin{gathered} p_0=(1,0,0,0),\ p_1=(0,1,0,0),\ p_2=(0,0,1,0),\ p_3=(0,0,0,1)\\ q_0=(-1,1,1,1),\ q_1=(1,-1,1,1),\ q_2=(1,1,-1,1),\ q_3=(1,1,1,-1)\end{gathered} \tag{9.13}$$

The points q_j belong to the torus $\mathbb{T}_A$ of (9.7), they correspond to the orbit $W(P)=(PQRS)$.
We extend the generic definition (9.10) to arbitray $\mathbf{u}\in P_3(\mathbb{C})\backslash S$ and define $F_{\mathbf{u}}$ in general as the union of S with the fiber $F(\mathbf{u})$ of Φ through $\mathbf{u}$. It can be understood geometrically as follows.
Let $\mathcal{N}$ be the net of quadrics in $P_3(\mathbb{C})$ which contain S. Given $\mathbf{u}\in P_3(\mathbb{C})\backslash S$ the elements of $\mathcal{N}$ which contain $\mathbf{u}$ form a pencil of quadrics with base locus

$$\cap\{Q\,|\,Q\in\mathcal{N},\,\mathbf{u}\in Q\}=Y_{\mathbf{u}} \tag{9.14}$$

which is an elliptic curve of degree 4 containing S and $\mathbf{u}$. One has

$$Y_{\mathbf{u}}=F_{\mathbf{u}} \tag{9.15}$$

With $\Phi(\mathbf{u})=(s_1,s_2,s_3)$ an explicit isomorphism of the elliptic curve $F_{\mathbf{u}}$ with the elliptic curve

$$E_3=\ \{(X,Y)\,|\,Y^2=\ \prod(Xs_j-1)\}$$

of (9.2) is given by

$$(X,Y)\to\mathbf{u}\,,\quad (u_k-u_0)(X\,s_k-1)-i\,(u_0+u_k)\,Y\ =\ 0\,,\quad \forall k\,. \tag{9.16}$$

Under this isomorphism the point at infinity in E_3 maps to $q_0\in F_{\mathbf{u}}$.

9.2. **Notations for ϑ-functions.**

Let us fix our notations for elliptic ϑ-functions. We fix $\tau\in\mathbb{H}$ a complex number of stricly positive imaginary part and let $q=e^{\pi i\tau}$ so that $|q|<1$. The basic ϑ-function is

$$\vartheta_3(z)=\sum_{\mathbb{Z}}q^{n^2}\,e^{2\pi i n z} \tag{9.17}$$

It is periodic in z with period 1 and is (up to scale) the only holomorphic section on $E=\mathbb{C}/L$, $L=\mathbb{Z}+\tau\mathbb{Z}$ of the line bundle associated to the periodicity conditions

$$\xi(z+1)=\ \xi(z)\,,\quad \xi(z+\tau)=\ q^{-1}\,e^{-2\pi i z}\,\xi(z)$$

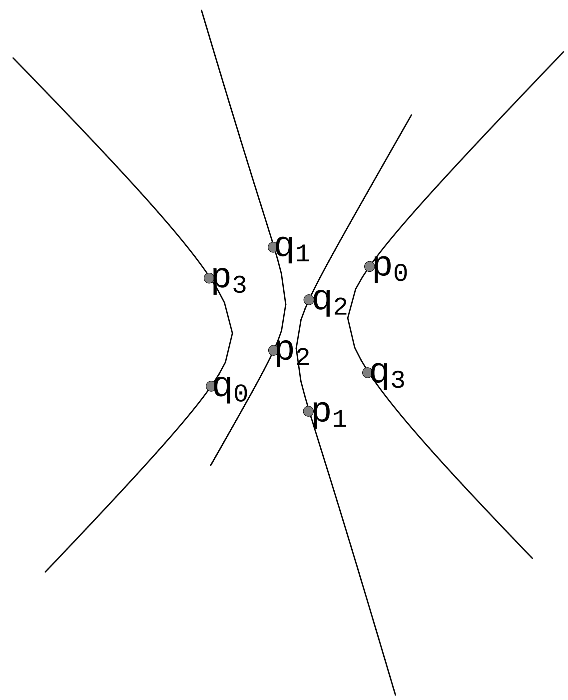

FIGURE 6. The Elliptic Curve $F_{\mathbf{u}} \cap P_3(\mathbb{R})$

In particular it is equal (up to scale) to the infinite product

$$\prod_1^\infty (1+q^{2n-1}\,e^{2\pi iz})(1+q^{2n-1}\,e^{-2\pi iz}),$$

and only vanishes at $\omega_2 = \frac{1}{2}+\frac{\tau}{2}$ modulo L.
The three other ϑ-functions are deduced from $\vartheta_3(z)$ by the translations of order two of E, more precisely one lets,

$$i\vartheta_1(z) = q^{\frac{1}{4}}\,e^{\pi iz}\,\vartheta_3(z+\omega_2)\,, \quad \vartheta_2(z) = \vartheta_1(z+\frac{1}{2})\,, \quad \vartheta_4(z) = \vartheta_3(z+\frac{1}{2})\,. \tag{9.18}$$

They all are holomorphic sections on $E = \mathbb{C}/L$, $L = \mathbb{Z}+\tau\mathbb{Z}$ of the line bundles associated to the periodicity conditions of the form

$$\xi(z+1) = \pm\,\xi(z)\,, \quad \xi(z+\tau) = \pm\,q^{-1}\,e^{-2\pi iz}\,\xi(z)$$

and it follows that the linear span of their squares $\vartheta_j^2(z)$ is a complex vector space of dimension two since all are holomorpic sections of a line bundle of degree two. It follows in particular that given any two theta functions the square of any other is a linear combination of the squares of the first two. The relevant coefficients are easy to compute from the values $\vartheta_j^2(0)$ and if one lets

$$k = \frac{\vartheta_2^2(0)}{\vartheta_3^2(0)}\,, \quad k' = \frac{\vartheta_4^2(0)}{\vartheta_3^2(0)}\,,$$

one gets

$$\vartheta_4^2(z) = k\,\vartheta_1^2(z) + k'\,\vartheta_3^2(z)\,, \quad \vartheta_2^2(z) = -\,k'\,\vartheta_1^2(z) + k\,\vartheta_3^2(z)\,, \quad \vartheta_1^2(z) = k\,\vartheta_4^2(z) - k'\,\vartheta_2^2(z)\,. \tag{9.19}$$

The λ-function of jacobi is given by $\lambda(\tau) = k^2$ and the $\wp$-function of Weierstrass by

$$\wp(z) = \alpha \frac{\vartheta_4^2(z)}{\vartheta_1^2(z)} + \beta\,,\ . \tag{9.20}$$

up to irrelevant normalization constants α and β. The main identities we shall use for ϑ-functions are the sixteen theta relations recalled in Appendix 3.

9.3. **Fiber = characteristic variety, and the birational automorphism σ of $\mathbb{P}_3(\mathbb{C})$.**

We shall now give, for generic values of (a, b, c) a parametrization of $F_{\mathbf{u}}$ by ϑ-functions. We start with the equations for $F_{\mathbf{u}}$

$$\frac{(u_0+u_1)(u_2+u_3)}{a} = \frac{(u_0+u_2)(u_3+u_1)}{b} = \frac{(u_0+u_3)(u_1+u_2)}{c} \tag{9.21}$$

and we diagonalize the above quadratic forms as follows

$$\begin{array}{rcl} (u_0+u_1)\,(u_2+u_3) & = & Z_0^2 - Z_1^2 \\ (u_0+u_2)\,(u_3+u_1) & = & Z_0^2 - Z_2^2 \\ (u_0+u_3)\,(u_1+u_2) & = & Z_0^2 - Z_3^2 \end{array} \tag{9.22}$$

where

$$(Z_0, Z_1, Z_2, Z_3) = M.u \tag{9.23}$$

where M is the involution,

$$M := \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{9.24}$$

In these terms the equations for $F_{\mathbf{u}}$ read

$$\frac{Z_0^2 - Z_1^2}{a} = \frac{Z_0^2 - Z_2^2}{b} = \frac{Z_0^2 - Z_3^2}{c} \tag{9.25}$$

Let now $\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$ and $\eta \in \mathbb{C}$ be such that one has, modulo projective equivalence,

$$(a, b, c) \sim \left(\frac{\vartheta_2(0)^2}{\vartheta_2(\eta)^2}, \frac{\vartheta_3(0)^2}{\vartheta_3(\eta)^2}, \frac{\vartheta_4(0)^2}{\vartheta_4(\eta)^2}\right) \tag{9.26}$$

where $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$ are the theta functions associated as above to τ.

More precisely let (a, b, c) be distinct non-zero complex numbers and

$$\lambda = \frac{a}{b}\,\frac{c-b}{c-a}\,, \quad p = \frac{a}{a-c} \tag{9.27}$$

and let $\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$ such that $\lambda(\tau) = \lambda$ where λ is Jacobi's λ function (*cf.* subsection 6.2). Let then $\eta \in \mathbb{C}$ be such that with $\omega_1 = \frac{1}{2}$, $\omega_3 = \frac{\tau}{2}$, one has

$$\frac{\wp(\eta) - \wp(\omega_3)}{\wp(\omega_1) - \wp(\omega_3)} = p\,,$$

where $\wp$ is the Weierstrass $\wp$-function for the lattice $L = \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$. This last equality does determine η only up to sign and modulo the lattice L but this ambiguity does not affect the validity of (9.26) which one checks using the basic properties of ϑ-functions recalled in subsection 9.2. Indeed in these terms one has $\lambda(\tau) = k^2$ which gives the first equality in (9.26) with (a, b, c) replaced by the right hand side of (9.26). Using (9.20) one gets the second equality since for any z one has

$$\frac{\wp(z) - \wp(\omega_3)}{\wp(\omega_1) - \wp(\omega_3)} = k\,\frac{\vartheta_4(z)^2}{\vartheta_1(z)^2}\,.$$

Let then τ and η be fixed by the above conditions, one gets

Proposition 9.3. *The following define isomorphisms of $\mathbb{C}/L$ with $F_{\mathbf{u}}$,*

$$\varphi(z) = \left(\frac{\vartheta_1(2z)}{\vartheta_1(\eta)}, \frac{\vartheta_2(2z)}{\vartheta_2(\eta)}, \frac{\vartheta_3(2z)}{\vartheta_3(\eta)}, \frac{\vartheta_4(2z)}{\vartheta_4(\eta)}\right) = (Z_0, Z_1, Z_2, Z_3)$$

and $\psi(z) = \varphi(z - \eta/2)$.

Proof. Up to an affine transformation, φ (and ψ are) is the classical projective embedding of $\mathbb{C}/L$ in $P_3(\mathbb{C})$. Thus we only need to check that the biquadratic curve $\operatorname{Im}\varphi = \operatorname{Im}\psi$ is given by (9.25). It is thus enough to check (9.25) on $\varphi(z)$. This follows from the basic relations (9.19) written as

$$\vartheta_3^2(z)\vartheta_2^2(0) = \vartheta_2^2(z)\vartheta_3^2(0) + \vartheta_1^2(z)\vartheta_4^2(0) \tag{9.28}$$

and

$$\vartheta_4^2(z)\vartheta_3^2(0) = \vartheta_1^2(z)\vartheta_2^2(0) + \vartheta_3^2(z)\vartheta_4^2(0) \tag{9.29}$$

which one uses to check $\frac{Z_0^2 - Z_1^2}{a} = \frac{Z_0^2 - Z_2^2}{b}$ and $\frac{Z_0^2 - Z_2^2}{b} = \frac{Z_0^2 - Z_3^2}{c}$ respectively.
Let us check the first one using (9.28) to replace all occurences of $\vartheta_3^2(2z)$ and $\vartheta_3^2(\eta)$ by the value given by (9.28). One gets

$$\frac{Z_0^2 - Z_2^2}{b} = \frac{\vartheta_1^2(2z)}{\vartheta_1^2(\eta)}\frac{\vartheta_3^2(\eta)}{\vartheta_3^2(0)} - \frac{\vartheta_3^2(2z)}{\vartheta_3^2(0)} = \frac{\vartheta_1^2(2z)}{\vartheta_1^2(\eta)}\frac{\vartheta_2^2(\eta)}{\vartheta_2^2(0)} + \frac{\vartheta_1^2(2z)\vartheta_4^2(0)}{\vartheta_2^2(0)\vartheta_3^2(0)} - (\frac{\vartheta_2^2(2z)}{\vartheta_2^2(0)} + \frac{\vartheta_1^2(2z)\vartheta_4^2(0)}{\vartheta_2^2(0)\vartheta_3^2(0)})$$

$$= \frac{\vartheta_1^2(2z)}{\vartheta_1^2(\eta)}\frac{\vartheta_2^2(\eta)}{\vartheta_2^2(0)} - \frac{\vartheta_2^2(2z)}{\vartheta_2^2(0)} = \frac{Z_0^2 - Z_1^2}{a}$$

Let us check the second one using (9.29) to replace all occurences of $\vartheta_4^2(2z)$ and $\vartheta_4^2(\eta)$ by the value given by (9.29). One gets

$$\frac{Z_0^2 - Z_3^2}{c} = \frac{\vartheta_1^2(2z)}{\vartheta_1^2(\eta)}\frac{\vartheta_4^2(\eta)}{\vartheta_4^2(0)} - \frac{\vartheta_4^2(2z)}{\vartheta_4^2(0)} = \frac{\vartheta_1^2(2z)}{\vartheta_1^2(\eta)}\frac{\vartheta_3^2(\eta)}{\vartheta_3^2(0)} + \frac{\vartheta_1^2(2z)\vartheta_2^2(0)}{\vartheta_3^2(0)\vartheta_4^2(0)} - (\frac{\vartheta_3^2(2z)}{\vartheta_3^2(0)} + \frac{\vartheta_1^2(2z)\vartheta_2^2(0)}{\vartheta_3^2(0)\vartheta_4^2(0)})$$

$$\frac{\vartheta_1^2(2z)}{\vartheta_1^2(\eta)}\frac{\vartheta_3^2(\eta)}{\vartheta_3^2(0)} - \frac{\vartheta_3^2(2z)}{\vartheta_3^2(0)} = \frac{Z_0^2 - Z_2^2}{b}$$

□

The elements of S are obtained from the following values of z

$$\psi(\eta) = p_0,\ \psi(\eta + \frac{1}{2}) = p_1,\ \psi(\eta + \frac{1}{2} + \frac{\tau}{2}) = p_2,\ \psi(\eta + \frac{\tau}{2}) = p_3 \tag{9.30}$$

and

$$\psi(0) = q_0,\ \psi(\frac{1}{2}) = q_1,\ \psi(\frac{1}{2} + \frac{\tau}{2}) = q_2,\ \psi(\frac{\tau}{2}) = q_3. \tag{9.31}$$

(We used $M^{-1}.\psi$ to go back to the coordinates u_μ but note that $M^{-1} = M$ and $M(q_j) = q_j$).
Let $H \sim \mathbb{Z}_2 \times \mathbb{Z}_2$ be the Klein subgroup of the symmetric group $\mathfrak{S}_4$ acting on $P_3(\mathbb{C})$ by permutation of the coordinates (u_0, u_1, u_2, u_3).
For ρ in H one has $\Phi \circ \rho = \Phi$, so that ρ defines for each $\mathbf{u}$ an automorphism of $F_{\mathbf{u}}$. For ρ in H the matrix $M\rho M^{-1}$ is diagonal with ± 1 on the diagonal and the quasiperiodicity of the ϑ-functions

allows to check that these automorphisms are translations on $F_{\mathbf{u}}$ by the following 2-torsion elements of $\mathbb{C}/L$,

$$\rho = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \end{pmatrix} \quad \text{is translation by}\, \omega_1 = \tfrac{1}{2},$$

(9.32)

$$\rho = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \end{pmatrix} \quad \text{is translation by}\, \omega_2 = \tfrac{1}{2} + \tfrac{\tau}{2}$$

Let $\mathcal{O} \subset P_3(\mathbb{C})$ be the complement of the 4 hyperplanes $\{u_\mu = 0\}$ with $\mu \in \{0,1,2,3\}$. Then $(u_0, u_1, u_2, u_3) \mapsto (u_0^{-1}, u_1^{-1}, u_2^{-1}, u_3^{-1})$ defines an involutive automorphism I of $\mathcal{O}$ and since one has

$$(u_0^{-1} + u_k^{-1})(u_\ell^{-1} + u_m^{-1}) = (u_0 u_1 u_2 u_3)^{-1}(u_0 + u_k)(u_\ell + u_m) \tag{9.33}$$

it follows that $\Phi \circ I = \Phi$, so that I defines for each $\mathbf{u} \in \mathcal{O}\backslash\{q_0, q_1, q_2, q_3\}$ an involutive automorphism of $F_{\mathbf{u}} \cap \mathcal{O}$ which extends canonically to $F_{\mathbf{u}}$. Note in fact that, as a birational map I continues to make sense on the complement of the 6 lines $\ell_{\mu,\nu} = \{u \,|\, u_\mu = 0,\, u_\nu = 0\}$ for $\mu, \nu \in \{0,1,2,3\}$ using $(u_0, u_1, u_2, u_3) \mapsto (u_1u_2u_3, u_0u_2u_3, u_0u_1u_3, u_0u_1u_2)$.

Proposition 9.4. *The restriction of I to $F_{\mathbf{u}}$ is the symmetry $\psi(z) \mapsto \psi(-z)$ around any of the points $q_\mu \in F_{\mathbf{u}}$ in the elliptic curve $F_{\mathbf{u}}$.*

This symmetry, as well as the above translations by two torsion elements does not refer to a choice of origin in the curve $F_{\mathbf{u}}$. The proof follows from identities on theta functions but it can be seen directly using the isomorphism $E_3 \simeq F_{\mathbf{u}}$ of (9.16). Indeed the symmetry around $q_0 \in F_{\mathbf{u}}$ corresponds to the transformation $(X, Y) \to (X, -Y)$ on E_3 and the isomorphism (9.16) carries this back to $(u_0, u_1, u_2, u_3) \mapsto (u_0^{-1}, u_1^{-1}, u_2^{-1}, u_3^{-1})$ as one checks directly dividing each of the equations (9.16) $(u_k - u_0)(X\, s_k - 1) - i\,(u_0 + u_k)\, Y = 0$ by $u_0 u_k$ to get the same equation but with $-Y$ instead of Y for the u_j^{-1}.

The torus $\mathbb{T}_A$ of (9.7) gives a covering of the real moduli space $\mathcal{M}$. For $\mathbf{u} \in \mathbb{T}_A$, the point $\Phi(\mathbf{u})$ is real with projective coordinates

$$\Phi(\mathbf{u}) = (s_1, s_2, s_3)\,, \quad s_k := 1 + t_\ell\, t_m\,, \quad t_k := \tan(\varphi_k - \varphi_0) \tag{9.34}$$

The corresponding fiber $F_{\mathbf{u}}$ is stable under complex conjugation $\mathbf{v} \mapsto \overline{\mathbf{v}}$ and the intersection of $F_{\mathbf{u}}$ with the real moduli space is given by,

$$F_{\mathbb{T}}(\mathbf{u}) = F_{\mathbf{u}} \cap \mathbb{T}_A = \{\mathbf{v} \in F_{\mathbf{u}} | I(\mathbf{v}) = \overline{\mathbf{v}}\} \tag{9.35}$$

The curve $F_{\mathbf{u}}$ is defined over $\mathbb{R}$ and (9.35) determines its purely <u>imaginary</u> points. Note that $F_{\mathbb{T}}(\mathbf{u})$ (9.35) is invariant under the Klein group H and thus has two connected components, we let $F_{\mathbb{T}}(\mathbf{u})^0$ be the component containing q_0. The real points, $\{\mathbf{v} \in F_{\mathbf{u}} | \mathbf{v} = \overline{\mathbf{v}}\} = F_{\mathbf{u}} \cap P_3(\mathbb{R})$ of $F_{\mathbf{u}}$ do play a complementary role in the characteristic variety as we shall see below.

Our aim now is to show that for $\mathbf{u} \in P_3(\mathbb{C})$ generic, there is an astute choice of generators of the quadratic algebra $\mathcal{A}_{\mathbf{u}} = C_{\mathrm{alg}}(\mathbb{C}^4(\mathbf{u}))$ for which the characteristic variety $E_{\mathbf{u}}$ actually coincides with the fiber variety $F_{\mathbf{u}}$ and to identify the corresponding automorphism σ. Since this coincidence no longer holds for non-generic values it is a quite remarkable fact which we first noticed by comparing the j-invariants of these two elliptic curves.

Let $\mathbf{u} \in P_3(\mathbb{C})$ be generic, we perform the following change of generators

$$\begin{aligned} y_0 &= \sqrt{u_1-u_2}\,\sqrt{u_2-u_3}\,\sqrt{u_3-u_1}\quad Y_0 \\ y_1 &= \sqrt{u_0+u_2}\,\sqrt{u_2-u_3}\,\sqrt{u_0+u_3}\quad Y_1 \\ y_2 &= \sqrt{u_0+u_3}\,\sqrt{u_3-u_1}\,\sqrt{u_0+u_1}\quad Y_2 \\ y_3 &= \sqrt{u_0+u_1}\,\sqrt{u_1-u_2}\,\sqrt{u_0+u_2}\quad Y_3 \end{aligned} \tag{9.36}$$

We let $J_{\ell m}$ be as before, given by (9.9)

$$J_{12} = \frac{a-b}{c},\;\; J_{23} = \frac{b-c}{a},\;\; J_{31} = \frac{c-a}{b} \tag{9.37}$$

with a, b, c given by (9.11). Finally let e_ν be the 4 points of $P_3(\mathbb{C})$ whose homogeneous coordinates (Z_μ) all vanish but one.

Theorem 9.5. *1) In terms of the Y_μ, the relations of $\mathcal{A}_{\mathbf{u}}$ take the form*

$$[Y_0, Y_k]_- = [Y_\ell, Y_m]_+ \tag{9.38}$$

$$[Y_\ell, Y_m]_- = -J_{\ell m}[Y_0, Y_k]_+ \tag{9.39}$$

for any $k \in \{1,2,3\}$, (k,ℓ,m) being a cyclic permutation of (1,2,3)

2) The characteristic variety $E_{\mathbf{u}}$ is the union of $F_{\mathbf{u}}$ with the 4 points e_ν.

3) The automorphism σ of the characteristic variety $E_{\mathbf{u}}$ is given by

$$\psi(z) \mapsto \psi(z-\eta) \tag{9.40}$$

on $F_{\mathbf{u}}$ and $\sigma = Id$ on the 4 points e_ν.

4) The automorphism σ is the restriction to $F_{\mathbf{u}}$ of a birational automorphism of $P_3(\mathbb{C})$ independent of $\mathbf{u}$ and defined over $\mathbb{Q}$.

The similarity between the above presentation and the Sklyanin one (cf. (4.10), (4.12)) is misleading, indeed for the latter all the characteristic varieties are contained in the same quadric (cf. [27] §2.4)

$$\sum x_\mu^2 = 0$$

and cant of course form a net of essentially disjoint curves.

Proof By construction $E_{\mathbf{u}} = \{Z \,|\, \text{Rank}\, N(Z) < 4\}$ where

$$N(Z) = \begin{pmatrix} Z_1 & -Z_0 & Z_3 & Z_2 \\ Z_2 & Z_3 & -Z_0 & Z_1 \\ Z_3 & Z_2 & Z_1 & -Z_0 \\ (b-c)Z_1 & (b-c)Z_0 & -aZ_3 & aZ_2 \\ (c-a)Z_2 & bZ_3 & (c-a)Z_0 & -bZ_1 \\ (a-b)Z_3 & -cZ_2 & cZ_1 & (a-b)Z_0 \end{pmatrix} \tag{9.41}$$

One checks that it is the union of the fiber $F_{\mathbf{u}}$ (in the generic case) with the above 4 points. In fact in terms of the original presentation (9.5) *i.e.* in terms of the y_j the characteristic variety in the generic case is the intersection of the two quadrics

$$\sum y_j^2 = 0\,, \quad \sum u_j^2\, y_j^2 = 0\,, \tag{9.42}$$

and after the change of variables (9.36) it just becomes

$$\frac{Z_0^2 - Z_1^2}{a} = \frac{Z_0^2 - Z_2^2}{b} = \frac{Z_0^2 - Z_3^2}{c} \tag{9.43}$$

as can easily be checked since only the squares Z_j^2 are involved (and linearly). Note that we already knew that the fiber $F_{\mathbf{u}}$ is abstractly isomorphic to the elliptic curve of the characteristic variety using lemma 6.2, proposition 9.2 and the isomorphism (9.16). But here we have shown that their respective embeddings in $\mathbb{P}^3(\mathbb{C})$ are the same (*i.e.* the corresponding line bundles are the same).
The automorphism σ of the characteristic variety $E_{\mathbf{u}}$ is given by definition by the equation,

$$N(Z)\,\sigma(Z) = 0 \tag{9.44}$$

where $\sigma(Z)$ is the column vector $\sigma(Z_\mu) := M \cdot \sigma(\mathbf{u})$ (in the variables Z_λ). One checks that $\sigma(Z)$ is already determined by the equations in (9.44) corresponding to the first three lines in $N(Z)$ which are independent of a, b, c (see below). Thus σ is in fact an automorphism of $P_3(\mathbb{C})$ which is the identity on the above four points and which restricts as automorphism of $F_{\mathbf{u}}$ for each $\mathbf{u}$ generic. One checks that σ is the product of two involutions which both restrict to $E_{\mathbf{u}}$ (for $\mathbf{u}$ generic)

$$\sigma = I \circ I_0 \tag{9.45}$$

where I is the involution of proposition 9.4 corresponding to $u_\mu \mapsto u_\mu^{-1}$ and where I_0 is given by

$$I_0(Z_0) = -Z_0,\ \ I_0(Z_k) = Z_k \tag{9.46}$$

for $k \in \{1, 2, 3\}$ and which restricts obviously to $E_{\mathbf{u}}$ in view of (9.22). Both I and I_0 are the identity on the above four points and since I_0 induces the symmetry $\varphi(z) \mapsto \varphi(-z)$ around $\varphi(0) = \psi(\eta/2)$ (proposition 9.3) one gets the result using proposition 9.4.

The fact that σ does not depend on the parameters a, b, c plays an important role. Explicitly we get from the first 3 equations (9.44)

$$\sigma(Z)_\mu = \eta_{\mu\mu}(Z_\mu^3 - Z_\mu \sum_{\nu\neq\mu} Z_\nu^2 - 2 \prod_{\lambda\neq\mu} Z_\lambda) \tag{9.47}$$

for $\mu \in \{0, 1, 2, 3\}$, where $\eta_{00} = 1$ and $\eta_{nn} = -1$ for $n \in \{1, 2, 3\}$. $\square$

Remark : Notice that, since the u_μ are homogeneous coordinates on $P_3(\mathbb{C})$, the generators y_μ as well as the generators Y_μ are only defined modulo a non zero multiplicative scalar. Indeed under a change $u_\mu \mapsto \nu^2 u_\mu$ of homogeneous coordinates ($\nu \in \mathbb{C}\backslash\{0\}$), the y_μ transform as $y_\mu \mapsto \nu^{-1} y_\mu$ while the Y_μ transform as $Y_\mu \mapsto \nu^{-4} Y_\mu$ which leaves invariant (9.6) and (9.36) as well as the x^μ ($(x^\mu)^2 = u_\mu (y_\mu)^2$). Later on, it will be convenient to choose a normalization for the generators, e.g. in (11.32).

10. The map from $\mathbb{T}^2_\eta \times [0, \tau]$ to S^3_φ and the pairing

This section contains the main technical result of the paper *i.e.* both the construction of the one-parameter family of $*$-homomorphisms from the algebra of S^3_φ (with φ generic) to the algebra of the non-commutative torus $\mathbb{T}^2_\eta$ and the computation of the pairing of the image of the Hochschild 3-cycle with the natural "fundamental class" for the product $\mathbb{T}^2_\eta \times [0, \tau]$.

10.1. **Central elements.**

We already saw in Section 2 (Lemma 2.1) that the algebra $C_{\text{alg}}(\mathbb{R}^4_\varphi)$ of $\mathbb{R}^4_\varphi$ contains in its center the following element,

$$Q_1 = C = \sum (x^\mu)^2 \,. \tag{10.1}$$

To get another one, one first looks at the Sklyanin algebra defined by (4.10) and (4.12) whose center contains two natural "Casimir" elements C_j,

$$C_1 = \sum S_\mu^2 \,, \quad C_2 = \sum j_k \, S_k^2 \,, \tag{10.2}$$

where the j_k fulfill the relations

$$-\frac{j_\ell - j_m}{j_k} = J_{\ell m} \tag{10.3}$$

Let us check that C_1 commutes with S_0 and S_ℓ. The idea is to only use the relation (4.10) so that the $J_{\ell m}$ do not interfere with the computation. This means that one treats the commutators as follows:

$$[S_0, S_k^2] = (S_0 \, S_k + S_k \, S_0) \, S_k - S_k (S_0 \, S_k + S_k \, S_0) = i[S_k \, [S_\ell \, S_m]] \,.$$

Thus the sum over k gives 0 in view of the Jacobi identy.

$$[S_1, S_0^2] = (S_1 \, S_0 + S_0 \, S_1) \, S_0 - S_0 (S_1 \, S_0 + S_0 \, S_1) = -\, i(S_2 \, S_3 - S_3 \, S_2) \, S_0 + i \, S_0 (S_2 \, S_3 - S_3 \, S_2) \,.$$

$$[S_1, S_2^2] = [S_1, S_2] \, S_2 + S_2 \, [S_1, S_2] = i(S_0 \, S_3 + S_3 \, S_0) \, S_2 + i \, S_2 (S_0 \, S_3 + S_3 \, S_0)$$

$$[S_1, S_3^2] = [S_1, S_3] \, S_3 + S_3 \, [S_1, S_3] = -\, i(S_0 \, S_2 + S_2 \, S_0) \, S_3 - i \, S_3 (S_0 \, S_2 + S_2 \, S_0) \,.$$

One checks that the sum of these terms gives 0. Using cyclic permutations the commutation with S_k easily follows.

Remark : It is worth noticing here that the relation $[C_1, S_\nu] = 0$ can be written in the form [16]

$$\sum_\mu [S_\mu, [S_\mu, S_\nu]_+] = 0$$

where it becomes apparent that it is a super Lie algebra version of the relation defining the Yang-Mills algebra studied in [14]. As pointed out in [16] the relation (4.10) is the corresponding super analog of the self-duality relation and the fact that it implies $[C_1, S_\nu] = 0$ is the content of Lemma 1 in [16].

Using (4.9) we can then assert that in the generic case the following element is in the center,

$$(\prod \sin \varphi_k) \, (x^0)^2 + \sum_1^3 \cos(\varphi_k - \varphi_\ell) \cos(\varphi_k - \varphi_m) \sin\varphi_k \, (x^k)^2 \,. \tag{10.4}$$

Substracting (10.1) multiplied by $\prod \sin\varphi_k$ and using

$$\cos(\varphi_k - \varphi_\ell)\cos(\varphi_k - \varphi_m) - \sin\varphi_\ell \sin\varphi_m = \cos\varphi_k \cos(-\varphi_k + \varphi_\ell + \varphi_m)$$

we then get:

$$Q_2 = \frac{1}{2}\sum_1^3 \sin 2\varphi_k \cos(-\varphi_k + \varphi_\ell + \varphi_m)\,(x^k)^2 \tag{10.5}$$

as a central element.

Note that to get that (10.2) is central we did not use relation (4.12) and thus it holds irrespective of the finiteness of the $J_{\ell m}$. Thus the case $\delta(\varphi) \neq 0$ is entirely covered to show that (10.5) is central.

Proposition 10.1. 1) *Both Q_j are in the center of $C_{\rm alg}(\mathbb{R}^4_\varphi)$ for all values of φ.*
2) *Let $S_\mu = \lambda_\mu\, x^\mu$ as in (4.9) and λ such that $\lambda\, j_k = (-s_k + s_\ell + s_m)/s_\ell s_m$ then*

$$Q_1 = \frac{C_1 - \lambda\, C_2}{\prod \sin\varphi_k}\,, \quad Q_2 = \lambda C_2\,. \tag{10.6}$$

Proof. 1) We just have to check for Q and this will be done replacing it by (10.4) and using instead of (4.10) the three relations (2.24),

$$\sin\varphi_k\,[x^0, x^k]_+ = i\cos(\varphi_\ell - \varphi_m)\,[x^\ell, x^m]\,. \tag{10.7}$$

We just have to repeat the same proof as for the S_μ's making sure that any $[x^0, x^k]_+$ has a $\sin\varphi_k$ as coefficient and every $[x^\ell, x^m]$ a $\cos(\varphi_\ell - \varphi_m)$. For the commutator with x^0 this follows from the term $\sin\varphi_k\,(x^k)^2$ in (10.4). For the commutator with x^1 this follows from the terms $\sin\varphi_1\,(x^0)^2$ and $\cos(\varphi_1 - \varphi_k)\,(x^k)^2$, $k \neq 1$.
2) Note that the existence of λ follows from lemma 4.7 *i.e.* with $\tilde{s}_k = (-s_k + s_\ell + s_m)/s_\ell s_m$ the equality

$$-\frac{\tilde{s}_\ell - \tilde{s}_m}{\tilde{s}_k} = J_{\ell m}$$

We have already shown that $C_1 = \prod \sin\varphi_k\, Q_1 + Q_2$. It remains to check that $Q_2 = \lambda\, C_2$. Since $\lambda\, j_k = \tilde{s}_k$ this amounts to $\lambda_k^2\, \tilde{s}_k = \frac{1}{2}\sin 2\varphi_k \cos(-\varphi_k + \varphi_\ell + \varphi_m)$ *i.e.*

$$\tilde{s}_k = \frac{\cos\varphi_k \cos(-\varphi_k + \varphi_\ell + \varphi_m)}{\cos(\varphi_k - \varphi_\ell)\cos(\varphi_k - \varphi_m)}$$

which follows from the definition of $\tilde{s}_k = (-s_k + s_\ell + s_m)/s_\ell s_m$. □

In terms of the presentation of theorem 9.5 one gets,

Proposition 10.2. *Let $\mathcal{A}_{\mathbf{u}} = C_{\rm alg}(\mathbb{C}^4(\mathbf{u}))$ at generic $\mathbf{u}$, then the following three linearly dependent quadratic elements belong to the center of $\mathcal{A}_{\mathbf{u}}$,*

$$Q_m = J_{k\ell}\,(Y_0^2 + Y_m^2) + Y_k^2 - Y_\ell^2\,. \tag{10.8}$$

10.2. **The Hochschild cycle** $\mathrm{ch}_{3/2}(U)$**.**

With $z^k = e^{i\varphi_k}\, x^k$ we have, with $\varphi_0 = 0$,

$$U = \sum \tau_\mu\, z^\mu\,, \qquad \tau_0 = 1\,, \quad \tau_k = i\,\sigma_k\,. \tag{10.9}$$

We need to compute

$$\mathrm{ch}_{3/2}(U) = U_{i_0 i_1} \otimes U^*_{i_1 i_2} \otimes U_{i_2 i_3} \otimes U^*_{i_3 i_3} - U^*_{i_0 i_1} \otimes U_{i_1 i_2} \otimes U^*_{i_2 i_3} \otimes U_{i_3 i_0}\,. \tag{10.10}$$

The trace computation is given by:

Lemma 10.3. *One has*

$$\frac{1}{2}\,\mathrm{Trace}\,(\tau_\alpha\, \tau^*_\beta\, \tau_\gamma\, \tau^*_\delta) = \delta_{\alpha\beta}\, \delta_{\gamma\delta} + \delta_{\beta\gamma}\, \delta_{\delta\alpha} - \delta_{\alpha\gamma}\, \delta_{\beta\delta} + \varepsilon_{\alpha\beta\gamma\delta}\,.$$

Proof. If the set $\{\alpha,\beta,\gamma,\delta\}$ is $\{0,1,2,3\}$ we can assume $\alpha = 0$ or $\beta = 0$ by symmetry in $(\alpha,\beta,\gamma,\delta) \to (\gamma,\delta,\alpha,\beta)$. For $\alpha = 0$ we get $\frac{1}{2}\,\mathrm{Trace}\,(\tau_\beta\, \tau_\gamma\, \tau_\delta)$ since the two $-$ signs from τ^* cancell. This is cyclic and antisymmetric and gives for $(1,2,3)$ using $\sigma_1\, \sigma_2 = i\,\sigma_3$ the result $i^4 \times \frac{1}{2} \times 2 = 1$. For $\beta = 0$ we get

$$\frac{1}{2}\,\mathrm{Trace}\,(\tau_\alpha\, \tau_\gamma\, \tau^*_\delta) = -\frac{1}{2}\,\mathrm{Trace}\,(\tau_\alpha\, \tau_\gamma\, \tau_\delta) = -\,\varepsilon_{0\alpha\gamma\delta} = \varepsilon_{\alpha\beta\gamma\delta}\,.$$

If two of the elements $\alpha,\beta,\gamma,\delta$ are equal and the two others are different we get 0. Thus there are 3 cases $\alpha = \beta$, $\alpha = \gamma$, $\alpha = \delta$. One has $\tau_\alpha\, \tau^*_\alpha = 1$, thus if $\alpha = \beta$ we get 1. For $\alpha = \gamma$ we get $\frac{1}{2}\,\mathrm{Trace}\,(\tau_\alpha\, \tau^*_\beta\, \tau_\alpha\, \tau^*_\beta)$. The two $-$ signs cancell and give $\frac{1}{2}\,\mathrm{Trace}\,(\tau_\alpha\, \tau_\beta\, \tau_\alpha\, \tau_\beta)$. We can assume $\alpha \neq \beta$. If $0 \in \{\alpha,\beta\}$ we get $\frac{1}{2}\,\mathrm{Trace}\,(\tau_k^2) = -1$. If $0 \notin \{\alpha,\beta\}$, $\tau_\beta\, \tau_\alpha = -\,\tau_\alpha\, \tau_\beta$ and we get again -1. The case $\alpha = \delta$, $\beta = \gamma$ is as $\alpha = \beta$, $\gamma = \delta$. Finally if all indices are equal we get 1. $\square$

Proposition 10.4. *The Hochschild cycle*

$$\mathrm{ch}_{3/2}(U) \in HZ_3(C^\infty(S^3_\varphi))$$

is given (using $\varphi_0 = 0$ and up to a scalar factor) by

$$\mathrm{ch}_{3/2}(U) = \sum \varepsilon_{\alpha\beta\gamma\delta} \cos(\varphi_\alpha - \varphi_\beta + \varphi_\gamma - \varphi_\delta)\, x^\alpha \otimes x^\beta \otimes x^\gamma \otimes x^\delta$$
$$-i \sum_{\mu,\nu} \sin 2(\varphi_\mu - \varphi_\nu)\, x^\mu \otimes x^\nu \otimes x^\mu \otimes x^\nu\,.$$

Proof. The coefficient of $x^\alpha \otimes x^\beta \otimes x^\gamma \otimes x^\delta$ in $\frac{1}{2}\mathrm{ch}_{3/2}(U)$ is given by $\left(\times \frac{1}{2}\right)$

$$\frac{1}{2}\,\mathrm{Trace}\,(\tau_\alpha\, \tau^*_\beta\, \tau_\gamma\, \tau^*_\delta)\, e^{i(\varphi_\alpha - \varphi_\beta + \varphi_\gamma - \varphi_\delta)} - \frac{1}{2}\,\mathrm{Trace}\,(\tau^*_\alpha\, \tau_\beta\, \tau^*_\gamma\, \tau_\delta)\, e^{i(-\varphi_\alpha + \varphi_\beta - \varphi_\gamma + \varphi_\delta)}\,. \tag{10.11}$$

It is non zero only in the two cases (with cardinality denoted as $\#$),

$$\#\,\{\alpha,\beta,\gamma,\delta\} = 4\,, \quad \#\,\{\alpha,\beta,\gamma,\delta\} \leq 2$$

In the first case we get as coefficient of $x^\alpha \otimes x^\beta \otimes x^\gamma \otimes x^\delta$ the term

$$\varepsilon_{\alpha\beta\gamma\delta}\, e^{i(\varphi_\alpha - \varphi_\beta + \varphi_\gamma - \varphi_\delta)} - \varepsilon_{\beta\gamma\delta\alpha}\, e^{-i(\varphi_\alpha - \varphi_\beta + \varphi_\gamma - \varphi_\delta)} = 2\varepsilon_{\alpha\beta\gamma\delta} \cos(\varphi_\alpha - \varphi_\beta + \varphi_\gamma - \varphi_\delta)$$

since the cyclic permutation has signature -1.

In the second case the terms $\delta_{\alpha\beta}\,\delta_{\gamma\delta}\,e^{i(\varphi_\alpha-\varphi_\beta+\varphi_\gamma-\varphi_\delta)}$ and $\delta_{\beta\gamma}\,\delta_{\delta\alpha}\,e^{i(\varphi_\alpha-\varphi_\beta+\varphi_\gamma-\varphi_\delta)}$ are just $\delta_{\alpha\beta}\,\delta_{\gamma\delta}$ and $\delta_{\beta\gamma}\,\delta_{\delta\alpha}$ and they cancell with the terms coming from the second part of 10.10

$$-\,\delta_{\beta\gamma}\,\delta_{\delta\alpha}\,e^{-i(\varphi_\alpha-\varphi_\beta+\varphi_\gamma-\varphi_\delta)} = -\,\delta_{\beta\gamma}\,\delta_{\delta\alpha} \quad \text{and} \quad -\,\delta_{\gamma\delta}\,\delta_{\alpha\beta}\,e^{-i(\varphi_\alpha-\varphi_\beta+\varphi_\gamma-\varphi_\delta)} = -\,\delta_{\gamma\delta}\,\delta_{\alpha\beta}\,.$$

Thus one remains with the following terms:

$$-\,\delta_{\alpha\gamma}\,\delta_{\beta\delta}\,e^{i(\varphi_\alpha-\varphi_\beta+\varphi_\gamma-\varphi_\delta)} - (-\,\delta_{\beta\delta}\,\delta_{\gamma\alpha}\,e^{-i(\varphi_\alpha-\varphi_\beta+\varphi_\gamma-\varphi_\delta)}) = -2\,i\,\sin(2(\varphi_\alpha-\varphi_\beta))$$

which yield the second term in the proposition. □

We now use the rescaling (4.9) in the case $\delta(\varphi)\neq 0$ and rewrite $\mathrm{ch}_{3/2}$ in terms of the generators S_μ. We let

$$\Lambda = \prod_1^3 (\tan(\varphi_j)\,\cos(\varphi_k-\varphi_\ell)) \tag{10.12}$$

$$s_0 = 0\,,\quad s_j = 1+\tan\varphi_k\tan\varphi_l\,,\quad \forall j\in\{1,2,3\}\,. \tag{10.13}$$

and

$$n_0 = 0\,,\quad n_k = 1\,,\quad \forall k\in\{1,2,3\}\,. \tag{10.14}$$

Corollary 10.5. *One has*

$$\begin{aligned}\Lambda\,\mathrm{ch}_{3/2} = &-\sum_{\alpha,\beta,\gamma,\delta}\varepsilon_{\alpha\beta\gamma\delta}(n_\alpha-n_\beta+n_\gamma-n_\delta)(s_\alpha-s_\beta+s_\gamma-s_\delta)\,S_\alpha\otimes S_\beta\otimes S_\gamma\otimes S_\delta\\ &+2\,i\sum_{\mu,\nu}(-1)^{n_\mu-n_\nu}\,(s_\mu-s_\nu)\,S_\mu\otimes S_\nu\otimes S_\mu\otimes S_\nu\,.\end{aligned}$$

Proof. We write the above formula as $\Lambda\,\mathrm{ch}_{3/2} = -A+2iB$. We let the λ_μ be as in lemma 4.4 so that

$$\prod_0^3 \lambda_\mu = -\delta(\varphi) = -\prod_1^3(\sin\varphi_k\cos(\varphi_\ell-\varphi_m)) \tag{10.15}$$

and

$$\lambda_0^2 = \prod_1^3 \sin\varphi_j\,,\quad \lambda_k^2 = \sin\varphi_k\,\cos(\varphi_k-\varphi_\ell)\cos(\varphi_k-\varphi_m)\,. \tag{10.16}$$

One has $x^\mu = \frac{S_\mu}{\lambda_\mu}$ and thus in $\Lambda\,\mathrm{ch}_{3/2}$ the first terms of proposition 10.4 give

$$\varepsilon_{\alpha\beta\gamma\delta}\,\frac{\Lambda}{\prod\lambda_\mu}\cos(\varphi_\alpha-\varphi_\beta+\varphi_\gamma-\varphi_\delta)S_\alpha\otimes S_\beta\otimes S_\gamma\otimes S_\delta \qquad (\varphi_0=0)\,. \tag{10.17}$$

One has

$$\frac{\Lambda}{\prod\lambda_\mu} = -\,\frac{1}{\prod\cos\varphi_k}\,.$$

Thus the presence of the term $-A$ follows from the equality

$$\frac{\cos(\varphi_\alpha-\varphi_\beta+\varphi_\gamma-\varphi_\delta)}{\prod\cos\varphi_k} = (n_\alpha-n_\beta+n_\gamma-n_\delta)(s_\alpha-s_\beta+s_\gamma-s_\delta) \tag{10.18}$$

which we now check. We let $t_j = \tan \varphi_j$. One gets

$$\frac{\cos(\varphi_1 - \varphi_2 - \varphi_3)}{\prod \cos\varphi_k} = 1 - t_2\, t_3 + t_1\, t_3 + t_1\, t_2$$

and more generally for *any* permutation σ of $1, 2, 3$ one has

$$\frac{\cos(\varphi_{\sigma(1)} - \varphi_{\sigma(2)} - \varphi_{\sigma(3)})}{\prod \cos\varphi_k} = -s_{\sigma(1)} + s_{\sigma(2)} + s_{\sigma(3)}\,. \tag{10.19}$$

Let us prove (10.18). Since $(\alpha, \beta, \gamma, \delta)$ is a permutation of $(0, 1, 2, 3)$ one of the indices is 0.
For $\alpha = 0$ we get $\varphi_\alpha = 0$ and

$$\frac{\cos(\varphi_\alpha - \varphi_\beta + \varphi_\gamma - \varphi_\delta)}{\prod \cos\varphi_k} = -(s_0 - s_\beta + s_\gamma - s_\delta)$$

since $s_0 = 0$. But $(n_\alpha - n_\beta + n_\gamma - n_\delta) = -1$ so that (10.18) holds.

For $\beta = 0$ we get $\varphi_\beta = 0$ and

$$\frac{\cos(\varphi_\alpha - \varphi_\beta + \varphi_\gamma - \varphi_\delta)}{\prod \cos\varphi_k} = (s_\alpha - s_0 + s_\gamma - s_\delta)$$

since $s_0 = 0$. But $(n_\alpha - n_\beta + n_\gamma - n_\delta) = 1$ so that (10.18) holds.

For $\gamma = 0$ we get $\varphi_\gamma = 0$ and

$$\frac{\cos(\varphi_\alpha - \varphi_\beta + \varphi_\gamma - \varphi_\delta)}{\prod \cos\varphi_k} = -(s_\alpha - s_\beta + s_0 - s_\delta)$$

since $s_0 = 0$. But $(n_\alpha - n_\beta + n_\gamma - n_\delta) = -1$ so that (10.18) holds.

Finally for $\delta = 0$, $\varphi_\delta = 0$ and

$$\frac{\cos(\varphi_\alpha - \varphi_\beta + \varphi_\gamma - \varphi_\delta)}{\prod \cos\varphi_k} = (s_\alpha - s_\beta + s_\gamma - s_0)$$

since $s_0 = 0$. But $(n_\alpha - n_\beta + n_\gamma - n_\delta) = 1$ so that (10.18) holds and we checked it in all cases.

Let us now compute the contribution of the second terms of proposition 10.4. For $i, j \in \{1, 2, 3\}$ one has

$$-\frac{\Lambda}{2\lambda_i^2\,\lambda_j^2}\sin 2(\varphi_i - \varphi_j) = \; s_i - s_j\,. \tag{10.20}$$

Indeed say with $i = 2$, $j = 3$, one gets

$$\frac{\Lambda}{\lambda_2^2\,\lambda_3^2} = \frac{\sin\varphi_1}{\cos\varphi_1\cos\varphi_2\cos\varphi_3\cos(\varphi_2 - \varphi_3)}\,.$$

Thus

$$-\frac{\Lambda}{2\lambda_2^2\,\lambda_3^2}\sin 2(\varphi_2 - \varphi_3) = -\frac{\sin\varphi_1\sin(\varphi_2 - \varphi_3)}{\cos\varphi_1\cos\varphi_2\cos\varphi_3} = -t_1(t_2 - t_3) = (1 + t_1\, t_3) - (1 + t_1\, t_2) = s_2 - s_3\,.$$

Next let us check that

$$\frac{\Lambda}{2\lambda_0^2\,\lambda_k^2}\sin 2\varphi_k = \; s_k\,. \tag{10.21}$$

Say with $k = 1$ one has $\lambda_0^2 = \prod \sin\varphi_j$, $\lambda_1^2 = \sin\varphi_1\cos(\varphi_1 - \varphi_2)\cos(\varphi_1 - \varphi_3)$ and

$$\frac{\Lambda}{2\lambda_0^2\,\lambda_1^2}\sin 2\varphi_1 = \frac{\cos(\varphi_2 - \varphi_3)}{\cos\varphi_2\cos\varphi_3} = s_1\,.$$

We thus get in general,

$$\frac{\Lambda}{\lambda_\mu^2\,\lambda_\nu^2}\sin 2(\varphi_\mu - \varphi_\nu) = -(-1)^{n_\mu - n_\nu}\, 2(s_\mu - s_\nu)\,. \tag{10.22}$$

Indeed, for $\mu, \nu \in \{1,2,3\}$ this is (10.20). If both $\mu, \nu = 0$ both sides are 0. Now both sides are antisymmetric in μ, ν thus one can take $\nu = 0$, $\mu \in \{1,2,3\}$. Then $n_\mu - n_\nu = 1$ and the result follows from (10.21). □

10.3. **Elliptic parameters.**

Let $\varphi \in \mathbb{T}_A$ and using (9.26) let $\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$ and $\eta \in \mathbb{C}$ and $\lambda \in \mathbb{C}$ such that

$$\lambda\,(s_1, s_2, s_3) = \left(\frac{\vartheta_2(0)^2}{\vartheta_2(\eta)^2}, \frac{\vartheta_3(0)^2}{\vartheta_3(\eta)^2}, \frac{\vartheta_4(0)^2}{\vartheta_4(\eta)^2}\right) \tag{10.23}$$

We shall call the triplet (τ, η, λ) elliptic parameters for φ. They are not unique given φ but they determine uniquely the s_j and hence φ up to an overall sign.

Lemma 10.6. *With the above notations* (9.27) *is equivalent to*

$$(\tilde{s}_1, \tilde{s}_2, \tilde{s}_3) = \lambda \left(\frac{\vartheta_2(0)\,\vartheta_2(2\eta)}{\vartheta_2(\eta)^2}, \frac{\vartheta_3(0)\,\vartheta_3(2\eta)}{\vartheta_3(\eta)^2}, \frac{\vartheta_4(0)\,\vartheta_4(2\eta)}{\vartheta_4(\eta)^2}\right).$$

Proof. By lemma 4.7 we just need to check that with

$$(j_1, j_2, j_3) = \left(\frac{\vartheta_2(0)\,\vartheta_2(2\eta)}{\vartheta_2(\eta)^2}, \frac{\vartheta_3(0)\,\vartheta_3(2\eta)}{\vartheta_3(\eta)^2}, \frac{\vartheta_4(0)\,\vartheta_4(2\eta)}{\vartheta_4(\eta)^2}\right) \tag{10.24}$$

one has

$$(\tilde{j}_1, \tilde{j}_2, \tilde{j}_3) = \left(\frac{\vartheta_2(0)^2}{\vartheta_2(\eta)^2}, \frac{\vartheta_3(0)^2}{\vartheta_3(\eta)^2}, \frac{\vartheta_4(0)^2}{\vartheta_4(\eta)^2}\right). \tag{10.25}$$

The three equalities $j_k\,\tilde{j}_\ell + \tilde{j}_k\,j_\ell = 2$ follow from the following identities on ϑ-functions,

$$\vartheta_3(0)^2\,\vartheta_2(0)\,\vartheta_2(2\eta) = \vartheta_2(\eta)^2\,\vartheta_3(\eta)^2 - \vartheta_1(\eta)^2\,\vartheta_4(\eta)^2\,,$$

$$\vartheta_2(0)^2\,\vartheta_3(0)\,\vartheta_3(2\eta) = \vartheta_2(\eta)^2\,\vartheta_3(\eta)^2 + \vartheta_1(\eta)^2\,\vartheta_4(\eta)^2\,,$$

and similarly

$$\vartheta_4(0)^2\,\vartheta_3(0)\,\vartheta_3(2\eta) = \vartheta_3(\eta)^2\,\vartheta_4(\eta)^2 - \vartheta_1(\eta)^2\,\vartheta_2(\eta)^2\,,$$

$$\vartheta_3(0)^2\,\vartheta_4(0)\,\vartheta_4(2\eta) = \vartheta_3(\eta)^2\,\vartheta_4(\eta)^2 + \vartheta_1(\eta)^2\,\vartheta_2(\eta)^2\,,$$

and

$$\vartheta_2(0)^2\,\vartheta_4(0)\,\vartheta_4(2\eta) = \vartheta_2(\eta)^2\,\vartheta_4(\eta)^2 + \vartheta_1(\eta)^2\,\vartheta_3(\eta)^2\,,$$

$$\vartheta_4(0)^2\,\vartheta_2(0)\,\vartheta_2(2\eta) = \vartheta_2(\eta)^2\,\vartheta_4(\eta)^2 - \vartheta_1(\eta)^2\,\vartheta_3(\eta)^2\,.$$

□

This lemma allows to relate the above parameters with those used by Sklyanin and one has with the above notations ([26])

$$J_{12} = \frac{\vartheta_1(\eta)^2\vartheta_4(\eta)^2}{\vartheta_2(\eta)^2\vartheta_3(\eta)^2}\,, \quad J_{23} = \frac{\vartheta_1(\eta)^2\vartheta_2(\eta)^2}{\vartheta_3(\eta)^2\vartheta_4(\eta)^2}\,, \quad J_{31} = -\frac{\vartheta_1(\eta)^2\vartheta_3(\eta)^2}{\vartheta_2(\eta)^2\vartheta_4(\eta)^2}\,, \tag{10.26}$$

which follows from the definition (9.27) of the elliptic parameters together with (9.19).

10.4. **The sphere S^3_φ and the noncommutative torus $\mathbb{T}^2_\eta$.**

Let $\varphi \in A^\circ$ so that $\frac{\pi}{2} > \varphi_1 > \varphi_2 > \varphi_3 > 0$. We can then choose the elliptic parameters τ and η such that

$$\tau \in i\,\mathbb{R}_+\,, \quad \eta \in [0,1]\,. \tag{10.27}$$

Then the module $q = e^{i\tau} \in]0,1[$ and the ϑ functions $\vartheta_j(z)$ are all real functions *i.e.* fulfill

$$\vartheta_j(\bar z) = \overline{\vartheta_j(z)}\,, \quad \forall z \in \mathbb{C}\,. \tag{10.28}$$

In particular the last elliptic parameter λ determined by (9.27) fulfills $\lambda > 0$.
We shall explain in this section how to use the representations constructed by Sklyanin [26] to obtain $*$-homomorphisms from $C_{\rm alg}(S^3_\varphi)$ to the algebra

$$C^\infty(\mathbb{T}^2_\eta) = C^\infty(\mathbb{R}/\mathbb{Z}) \rtimes_\eta \mathbb{Z}\,, \tag{10.29}$$

obtained as the crossed product of the algebra $C^\infty(\mathbb{R}/\mathbb{Z})$ of smooth periodic functions by the translation η. Recall that a generic element of $C^\infty(\mathbb{T}^2_\eta)$ is of the form

$$f = \sum_{\mathbb{Z}} f_n\, V^n$$

while the basic algebraic rule is given by

$$V\,f\,V^{-1}(u) = f(u+\eta)\,, \quad \forall u \in \mathbb{R}/\mathbb{Z}\,, \quad \forall f \in C^\infty(\mathbb{R}/\mathbb{Z})\,. \tag{10.30}$$

Moreover $C^\infty(\mathbb{T}^2_\eta)$ is an involutive algebra with involution turning V into a unitary operator.

Starting from the representations constructed in [26] and conjugating by the operator

$$M(\xi)(u) = e^{-2\pi i u\, v/\eta}\, \xi(u + \tau/4)$$

one performs a shift in the indices of the ϑ-functions based on

$$\vartheta_1(z+\frac{\tau}{2}) = q^{-\frac14}\, e^{-\pi i z} i\vartheta_4(z)\,, \quad \vartheta_2(z+\frac{\tau}{2}) = q^{-\frac14}\, e^{-\pi i z}\vartheta_3(z)\,,$$
$$\vartheta_3(z+\frac{\tau}{2}) = q^{-\frac14}\, e^{-\pi i z}\vartheta_2(z)\,, \quad \vartheta_4(z+\frac{\tau}{2}) = q^{-\frac14}\, e^{-\pi i z} i\vartheta_1(z)\,.$$

which allows to replace the singular denominator $\vartheta_1(2u)$ by $\vartheta_4(2u)$ which no longer vanishes for $u \in \mathbb{R}/\mathbb{Z}$.
One obtains this way a homomorphism from the Sklyanin algebra to $C^\infty(\mathbb{T}^2_\eta)$ but it is not yet unitary and to make it so one needs to conjugate again by a multiplication operator of the form,

$$N(\xi)(u) = \bar d(u)\, \xi(u)$$

where the function $d \in C^\infty(\mathbb{R}/\mathbb{Z})$ fulfills the following conditions,

$$d(u)\bar d(u) = \vartheta_4(2u)\,, \quad \forall u \in \mathbb{R}/\mathbb{Z}\,.$$

We use the identity

$$\vartheta_3(0)^2\, \vartheta_4(0)\, \vartheta_4(2u) = \vartheta_3(u)^2\, \vartheta_4(u)^2 + \vartheta_1(u)^2\, \vartheta_2(u)^2\,,$$

and thus take

$$c\;\, d(u) = \vartheta_3(u)\, \vartheta_4(u) + i\, \vartheta_1(u)\, \vartheta_2(u)\,, \quad c^2 = \vartheta_3(0)^2\, \vartheta_4(0)\,. \tag{10.31}$$

Note that one has

$$\bar d(u) = d(-u)\,, \quad \forall u \in \mathbb{R}/\mathbb{Z}$$

The effect of the conjugacy $N\,.\,N^{-1}$ on simple monomials is the following

$$\frac{f(u)}{\vartheta_4(2\,u)}\,V \to \frac{f(u)}{d(u)\,d(-u-\eta)}\,V\,,\quad \frac{f(u)}{\vartheta_4(2\,u)}\,V^* \to \frac{f(u)}{d(u)\,d(-u+\eta)}\,V^*$$

The formulas which define the images $\rho(S_\alpha)$ then become, with $m \in [0,\tau]$,

$$\rho(S_0) = \vartheta_1(\eta)\,\frac{\vartheta_3(2u+\eta+im)}{d(u)\,d(-u-\eta)}\,V + \vartheta_1(\eta)\,\frac{\vartheta_3(2u-\eta-im)}{d(u)\,d(-u+\eta)}\,V^* \tag{10.32}$$

$$\rho(S_1) = -i\,\vartheta_2(\eta)\,\frac{\vartheta_4(2u+\eta+im)}{d(u)\,d(-u-\eta)}\,V + i\,\vartheta_2(\eta)\,\frac{\vartheta_4(2u-\eta-im)}{d(u)\,d(-u+\eta)}\,V^* \tag{10.33}$$

$$\rho(S_2) = \vartheta_3(\eta)\,\frac{\vartheta_1(2u+\eta+im)}{d(u)\,d(-u-\eta)}\,V + \vartheta_3(\eta)\,\frac{\vartheta_1(2u-\eta-im)}{d(u)\,d(-u+\eta)}\,V^* \tag{10.34}$$

$$\rho(S_3) = -\vartheta_4(\eta)\,\frac{\vartheta_2(2u+\eta+im)}{d(u)\,d(-u-\eta)}\,V - \vartheta_4(\eta)\,\frac{\vartheta_2(2u-\eta-im)}{d(u)\,d(-u+\eta)}\,V^*\,. \tag{10.35}$$

and one has

Theorem 10.7. *The formulas* (10.32)...(10.35) *define a $*$-homomorphism from the Sklyanin algebra to $C^\infty(\mathbb{T}^2_\eta)\widehat{\otimes}C^\infty([0,\tau])$.*

Proof. Since by construction ρ is conjugate to an homomorphism it is an homomorphism and we just need to check that the images $\rho(S_\mu)$ of the generators are self-adjoint elements of $C^\infty(\mathbb{T}^2_\eta)$ for each value of $m \in [0,\tau]$.
One has

$$\Big(\frac{f(u)}{d(u)\,d(-u-\eta)}\,V\Big)^* = V^*\,\frac{\bar f(u)}{d(-u)\,d(u+\eta)}$$

since $\eta \in \mathbb{R}$ and $\bar d(x) = d(-x)$ for $x \in \mathbb{R}$. Thus using (10.30) one gets

$$\Big(\frac{f(u)}{d(u)\,d(-u-\eta)}\,V\Big)^* = \frac{\bar f(u-\eta)}{d(-u+\eta)\,d(u)}\,V^*$$

Since m is real one has

$$\bar\vartheta_j(x+im) = \vartheta_j(x-im)\,,\quad \forall x \in \mathbb{R}\,,\quad \forall j$$

using (10.28). Thus one checks directly the required self-adjointness of the $\rho(S_\mu)$. ☐

To obtain a $*$-homomorphism from $C_{\rm alg}(S^3_\varphi)$ to $C^\infty(\mathbb{T}^2_\eta)\otimes C^\infty([0,\tau])$ we need to normalize the above formulas so that the element Q_1 in the center of $C_{\rm alg}(\mathbb{R}^4_\varphi)$ gets mapped to 1. By proposition 10.6 this amounts to introduce an overall scaling factor given by

$$\sigma(m) = \Big(\prod \sin\varphi_j\Big)^{1/2}\,(C_1 - \lambda\,C_2)^{-1/2}\,. \tag{10.36}$$

where the explicit values of the Casimirs C_j are given from [26] by

$$C_1 = 4\,\vartheta_2^2(im)\,,\quad C_2 = 4\,\vartheta_2(\eta+im)\,\vartheta_2(\eta-im)\,. \tag{10.37}$$

We can now normalize the above homomorphism ρ as

$$\tilde\rho(S_j) = \sigma(m)\,\rho(S_j)\,. \tag{10.38}$$

We then get using the change of variables

$$S_\mu = \lambda_\mu\,x^\mu\,,$$

Corollary 10.8. *The map $\tilde{\rho}$ defines a $*$-homomorphism*

$$C_{\mathrm{alg}}(S^3_\varphi) \to C^\infty(\mathbb{T}^2_\eta)\widehat{\otimes} C^\infty([0,\tau])\,.$$

Proof. Since $\varphi \in A$ one has $\lambda_\mu \in \mathbb{R}$ and the above change of variables is a $*$-isomorphism of $C_{\mathrm{alg}}(\mathbb{R}^4_\varphi)$ with the Sklyanin algebra. Thus we only need to check that with the above normalization the $*$-homomorphism $\tilde{\rho}$ maps the central element Q_1 which determines the sphere to the element

$$1 \in C^\infty(\mathbb{T}^2_\eta)\widehat{\otimes} C^\infty([0,\tau])$$

This follows from Equation (10.6). ☐

In the following we shall use the notation $C^\infty(\mathbb{T}^2_\eta \times [0,\tau])$ to denote the completed tensor product $C^\infty(\mathbb{T}^2_\eta)\widehat{\otimes} C^\infty([0,\tau])$.

10.5. **Pairing with** $[\mathbb{T}^2_\eta]$.

In order to test the non-triviality of the $*$-homomorphism $\tilde{\rho}$ we shall compute what will be later interpreted as its Jacobian. In order to do this we shall pair the image

$$\tilde{\rho}_*(\mathrm{ch}_{3/2}) \in HZ_3(C^\infty(\mathbb{T}^2_\eta \times [0,\tau])) \tag{10.39}$$

with the natural Hochschild three cocycle obtained using the fundamental class $[\mathbb{T}^2_\eta]$ introduced in [8]. Since the variable $m \in [0,\tau]$ labels the center we shall view the above pairing as defining a function of m.
The basic hochschild three cocycle on $C^\infty(\mathbb{T}^2_\eta \times [0,\tau]))$ is given by

$$\tau(a_0,\cdots,a_3) = \sum \epsilon_{ijk}\,\tau_0(a_0\,\delta_i(a_1)\,\delta_j(a_2)\,\delta_k(a_3))\ . \tag{10.40}$$

where τ_0 is the trace obtained as the tensor product of the canonical trace χ on $C^\infty(\mathbb{T}^2_\eta)$ by the trace on $C^\infty([0,\tau])$ given by integration,

$$\tau_0(a)) = \int_0^\tau \chi(a(m))dm\,, \quad \chi(f) = \int_0^1 f(u)du\,, \quad \chi(f\,V^n) = 0\,, \quad \forall n \neq 0\,. \tag{10.41}$$

The three basic derivations δ_j are given by

$$\delta_1 = \partial/\partial m\,, \quad \delta_2 = \partial/\partial u\,, \quad \delta_3 = 2\pi i\,V\,\partial/\partial V\,, \tag{10.42}$$

where in the last term the differentiation $V\,\partial/\partial V$ has the effect of multiplying by n any monomial $f\,V^n$.
As the product of τ by any function $h(m)$ viewed as an element of the center of $C^\infty(\mathbb{T}^2_\eta \times [0,\tau]))$ is still a Hochschild three cocycle we obtain a differential one form on $[0,\tau]$ as the pairing

$$\omega = <\mathrm{ch}_{3/2},\,\tau> \tag{10.43}$$

The basic lemma is then the following using the notation

$$(\sigma_0,\sigma_1,\sigma_2,\sigma_3) = \left(0,\,\frac{\vartheta_2(0)^2}{\vartheta_2(\eta)^2},\,\frac{\vartheta_3(0)^2}{\vartheta_3(\eta)^2},\,\frac{\vartheta_4(0)^2}{\vartheta_4(\eta)^2}\right) \tag{10.44}$$

Lemma 10.9. *With the above notations one has*

$$< \tau, \sum_{\alpha,\beta,\gamma,\delta} \varepsilon_{\alpha\beta\gamma\delta}(n_\alpha - n_\beta + n_\gamma - n_\delta)(\sigma_\alpha - \sigma_\beta + \sigma_\gamma - \sigma_\delta)\, \rho(S_\alpha) \otimes \rho(S_\beta) \otimes \rho(S_\gamma) \otimes \rho(S_\delta)$$

$$-2\, i \sum_{\mu,\nu} (-1)^{n_\mu - n_\nu} \, (\sigma_\mu - \sigma_\nu)\, \rho(S_\mu) \otimes \rho(S_\nu) \otimes \rho(S_\mu) \otimes \rho(S_\nu) > =$$

$$24\,(2\pi i)^3 \, \frac{\vartheta_1'(0)^3}{\pi^3} \, \frac{\vartheta_1(\eta)\, \vartheta_1(2im)}{\vartheta_2(\eta)\vartheta_3(\eta)\vartheta_4(\eta)} \; .$$

The precise meaning of the equality is that for any $h \in C^\infty([0,\tau]))$ the evaluation of the Hochschild cocycle τ on the product of the Hochschild cycle of the right hand side by h gives the integral

$$\int_0^\tau h(m)\, g(m)\, dm$$

where[6]

$$g(m) = \, 24\,(2\pi i)^3 \, \frac{\vartheta_1'(0)^3}{\pi^3} \, \frac{\vartheta_1(\eta)\, \vartheta_1(2im)}{\vartheta_2(\eta)\vartheta_3(\eta)\vartheta_4(\eta)} \; . \tag{10.45}$$

The proof of this is a long computation based on the "a priori" properties of the pairing which allow to show that the dependence in the parameters η and m is of the expected form, while the dependence in the module q is that of a modular form. It then follows from the explicit knowledge of enough terms in the q-expansion that the above formula is valid. So far we have not been able to eliminate completely the use of the computer to check this validity and its understanding will only come through the gradual simplifications below.

We shall now show that the dependance in m of the normalization factor $\sigma(m)$ in the definition (10.38) of the homomorphism $\tilde{\rho}$ can be ignored when one computes the pairing (10.43)

Lemma 10.10. *Let δ, δ' be derivations of the unital algebra $\mathcal{A}$ preserving a trace τ_0 on $\mathcal{A}$. Let ϕ_j be the multilinear forms on $\mathcal{A}$ given by*

$$\phi_1(a_0, a_1, a_2, a_3) = \, \tau_0(a_0\, a_1\, \delta(a_2)\, \delta'(a_3))\, , \quad \phi_2(a_0, a_1, a_2, a_3) = \, \tau_0(a_0\, \delta(a_1)\, a_2\, \delta'(a_3))$$

Then for any invertible $U \in \mathcal{A}$ one has

$$\phi_j(U, U^{-1}, U, U^{-1}) - \, \phi_j(U^{-1}, U, U^{-1}, U) = \, 0\, .$$

Proof. One has

$$\phi_1(U, U^{-1}, U, U^{-1}) = \, \tau_0(U\, U^{-1}\, \delta(U)\, \delta'(U^{-1})) = \, -\tau_0(\delta(U)\, U^{-1}\, \delta'(U)\, U^{-1})$$

$$\phi_1(U^{-1}, U, U^{-1}, U) = \, \tau_0(U^{-1}\, U\, \delta(U^{-1})\, \delta'(U)) = \, -\tau_0(U^{-1}\, \delta(U)\, U^{-1}\, \delta'(U))$$

thus the cyclicity of the trace proves the statement for $j = 1$. Similarly one has

$$\phi_2(U, U^{-1}, U, U^{-1}) = \, \tau_0(U\, \delta(U^{-1})\, U\, \delta'(U^{-1})) = \, \tau_0(\delta(U)\, U^{-1}\, \delta'(U)\, U^{-1})$$

$$\phi_2(U^{-1}, U, U^{-1}, U) = \, \tau_0(U^{-1}\, \delta(U)\, U^{-1}\, \delta'(U))$$

and the cyclicity of the trace proves the statement for $j = 2$. □

We thus get the following result

[6]The q-expansion of the fraction $\frac{\vartheta_1'(0)}{\pi}$ has rational coefficients.

Corollary 10.11. *The pairing of $\tilde{\rho}_*(\mathrm{ch}_{3/2})$ with τ is given by the differential form*

$$\omega = -\frac{\sigma(m)^4\, g(m)}{\lambda\,\Lambda}\, dm\,,$$

with Λ given in (10.12), *λ by* (9.27), *$\sigma(m)$ by* (10.36) *and $g(m)$ by* (10.45).

Proof. Using lemma 10.9 and corollary 10.5 one just needs to show that the terms of the form

$$\delta_1(\tilde{\rho}(S_j)) - \sigma(m)\,\delta_1(\rho(S_j)) = \frac{d\sigma(m)}{dm}\,\rho(S_j)$$

do not contribute. But their total contribution is a sum of six terms each of which is of the form

$$\phi_j(U, U^{-1}, U, U^{-1}) - \phi_j(U^{-1}, U, U^{-1}, U)$$

where $U \in M_2(C_{\mathrm{alg}}(S^3_\varphi))$ is the basic unitary element while ϕ_j is as in lemma 10.10 with $\delta, \delta' \in \{\delta_2, \delta_3\}$. Thus each of these terms vanishes by lemma 10.10. □

10.6. Simplifying the $*$-homomorphism $\tilde{\rho}$.

We shall make several simplifications in the formulas involved in the construction of the $*$-homomorphism $\tilde{\rho}$ of corollary 10.8 in order to gradually eliminate all ϑ-functions and express the result in purely algebraic terms.

The denominators involved in the construction of the $*$-homomorphism $\tilde{\rho}$ are of the form

$$d(u)\,d(-u \pm \eta) \tag{10.46}$$

where by (10.31),

$$c\ d(u) = \vartheta_3(u)\,\vartheta_4(u) + i\,\vartheta_1(u)\,\vartheta_2(u)\,, \quad c^2 = \vartheta_3(0)^2\,\vartheta_4(0)\,.$$

Our first task will be to rewrite (10.46) as a linear form in terms of the projective coordinates $\psi(u)$ of proposition 9.3 *i.e.*

$$\psi(u) = \left(\frac{\vartheta_1(2u-\eta)}{\vartheta_1(\eta)}, \frac{\vartheta_2(2u-\eta)}{\vartheta_2(\eta)}, \frac{\vartheta_3(2u-\eta)}{\vartheta_3(\eta)}, \frac{\vartheta_4(2u-\eta)}{\vartheta_4(\eta)}\right) = (Z_0, Z_1, Z_2, Z_3)$$

Lemma 10.12. *With the above notations one has*

$$\vartheta_3(0)\,d(u)\,d(-u+\eta) = i\,\vartheta_1(\eta)\vartheta_2(\eta)\,Z_1 + \vartheta_3(\eta)\vartheta_4(\eta)\,Z_3 \tag{10.47}$$

Proof. One has

$$\begin{aligned} c^2\,d(u)\,d(-u+\eta) &= (\vartheta_3(u)\,\vartheta_4(u) + i\,\vartheta_1(u)\,\vartheta_2(u))(\vartheta_3(u-\eta)\,\vartheta_4(u-\eta) - i\,\vartheta_1(u-\eta)\,\vartheta_2(u-\eta)) \\ &= \vartheta_3(u)\,\vartheta_4(u)\,\vartheta_3(u-\eta)\,\vartheta_4(u-\eta) + \vartheta_1(u)\,\vartheta_2(u)\,\vartheta_1(u-\eta)\,\vartheta_2(u-\eta) \\ &\quad + i\,\vartheta_1(u)\,\vartheta_2(u)\,\vartheta_3(u-\eta)\,\vartheta_4(u-\eta) - i\,\vartheta_3(u)\,\vartheta_4(u)\,\vartheta_1(u-\eta)\,\vartheta_2(u-\eta) \end{aligned}$$

Thus using the basic addition formulas (obtained from (15.6) and (15.15))

$$\vartheta_3(x)\,\vartheta_4(x)\,\vartheta_3(y)\,\vartheta_4(y) - \vartheta_1(x)\,\vartheta_2(x)\,\vartheta_1(y)\,\vartheta_2(y)) = \vartheta_3(0)\,\vartheta_4(0)\,\vartheta_3(x+y)\,\vartheta_4(x-y)$$

$$\vartheta_1(x)\,\vartheta_2(x)\,\vartheta_3(y)\,\vartheta_4(y) + \vartheta_3(x)\,\vartheta_4(x)\,\vartheta_1(y)\,\vartheta_2(y)) = \vartheta_3(0)\,\vartheta_4(0)\,\vartheta_1(x+y)\,\vartheta_2(x-y)$$

for $x = u$, $y = \eta - u$, we get

$$c^2\,d(u)\,d(-u+\eta) = \vartheta_3(0)\,\vartheta_4(0)\,(\vartheta_3(\eta)\,\vartheta_4(2u-\eta) + i\,\vartheta_1(\eta)\,\vartheta_2(2u-\eta))\,,$$

which gives the required equality. □

To simplify the numerators involved in the construction of the $*$-homomorphism $\tilde{\rho}$ we pass from generators S_μ of the Sklyanin algebra to the generators Y_μ of Theorem 9.5 by the following transformation

$$S_0 = d\,Y_2\,, \quad S_1 = i\,Y_3\,, \quad S_2 = d\,Y_0\,, \quad S_3 = -Y_1\,, \tag{10.48}$$

where $d = \dfrac{\vartheta_1(\eta)\vartheta_3(\eta)}{\vartheta_2(\eta)\vartheta_4(\eta)}$.
One checks that the Y_μ fulfill the presentation of Theorem 9.5 using the equality (10.26) $d^2 = -J_{31}$. We can then reformulate the construction of the homomorphism ρ in the following terms,

Lemma 10.13. *With the above notations one has, up to an overall scalar factor γ,*

$$\rho(Y_\mu) = \frac{\psi_\mu(u - i\,m/2)}{L(u)}\; V^* + \epsilon_\mu\, V\; \frac{\psi_\mu(u + i\,m/2)}{\bar{L}(u)} \tag{10.49}$$

where $\epsilon = (1,1,1,-1)$ and

$$L(u) = i\,\vartheta_1(\eta)\vartheta_2(\eta)\,\psi_1(u) + \vartheta_3(\eta)\vartheta_4(\eta)\,\psi_3(u)\,, \quad \bar{L}(u) = \overline{L(u)}\,.$$

Proof. One just needs to perform the transformation (10.48) on the equations (10.32)....(10.35). One gets an overall scalar factor

$$\gamma = \vartheta_2(\eta)\,\vartheta_4(\eta)\,\vartheta_3(0)\,.$$

multiplying the right hand side of (10.49) (or equivalently dividing $L(u)$). ∎

In order to understand (10.49) we let

$$Z = \psi(u - i\,m/2)\,, \quad Z' = \epsilon\,\psi(u + i\,m/2)\,, \quad W = L(u)^{-1}\,V^*\,, \quad W' = V\,\bar{L}(u)^{-1}\,. \tag{10.50}$$

Note that one has $Z' = \epsilon\,\bar{Z}$ and $W' = W^*$ but we shall ignore that for a while and treat for instance Z and Z' as independent variables. The multiplicative terms such as $L(u)^{-1}$ do not alter the cross product rules (10.30) but they alter the simplification rule $V\,V^* = V^*\,V = 1$. Our next task will thus be to give a simple expression for $W\,W'$ in terms of (Z, Z').
One has by construction

$$W\,W' = (L(u)\,\bar{L}(u))^{-1}\,, \tag{10.51}$$

and we need to express the denominator in terms of Z and Z'. Note that we have the freedom to multiply by an arbitrary function of m since this only alters the normalization of ρ which is needed in any case to pass to $\tilde{\rho}$.

Lemma 10.14. *With the above notations one has,*

$$\nu(m)\; L(u)\,\bar{L}(u) = J_{23}\,(Z_0\,Z_0' + Z_1\,Z_1') + Z_2\,Z_2' - Z_3\,Z_3'\,, \tag{10.52}$$

where

$$\nu(m) = \frac{2\,\vartheta_3^2(im)}{\vartheta_3^2(0)\vartheta_3^2(\eta)\vartheta_4^2(\eta)}\,.$$

Proof. One has

$$L(u)\,\bar{L}(u) = \vartheta_1^2(\eta)\vartheta_2^2(2u - \eta) + \vartheta_3^2(\eta)\vartheta_4^2(2u - \eta)\,,$$

thus with

$$a = 2u - \eta + im\,, \quad b = 2u - \eta - im\,, \quad \frac{a+b}{2} = 2u - \eta\,, \quad \frac{a-b}{2} = im$$

we get

$$\vartheta_3^2(im)\; L(u)\,\bar{L}(u) = \vartheta_1^2(\eta)\,\vartheta_3^2(\frac{a-b}{2})\,\vartheta_2^2(\frac{a+b}{2}) + \vartheta_3^2(\eta)\;\vartheta_3^2(\frac{a-b}{2})\,\vartheta_4^2(\frac{a+b}{2})\,. \tag{10.53}$$

We now use the addition formulas

$$2\,\vartheta_3^2(\frac{a-b}{2})\,\vartheta_2^2(\frac{a+b}{2}) = \vartheta_2^2(0)\,\vartheta_3(a)\,\vartheta_3(b) + \vartheta_3^2(0)\,\vartheta_2(a)\,\vartheta_2(b) - \vartheta_4^2(0)\,\vartheta_1(a)\,\vartheta_1(b)$$

(adding (15.9) and (15.10)) and

$$2\,\vartheta_3^2(\frac{a-b}{2})\,\vartheta_4^2(\frac{a+b}{2}) = \vartheta_2^2(0)\,\vartheta_1(a)\,\vartheta_1(b) + \vartheta_3^2(0)\,\vartheta_4(a)\,\vartheta_4(b) + \vartheta_4^2(0)\,\vartheta_3(a)\,\vartheta_3(b)$$

(adding (15.5) and (15.6)) which allow to write (10.53) as a symmetric bilinear form in (Z, Z'). One then uses (9.19) and (10.26)

$$J_{23} = \frac{\vartheta_1^2(\eta)\vartheta_2^2(\eta)}{\vartheta_3^2(\eta)\vartheta_4^2(\eta)}\,,$$

to obtain the required equality. □

Proposition 10.15. *With the above notations one has, up to an overall scalar factor $\delta(m)$,*

$$\rho(Y_\mu) = Z_\mu\; W + W'\; Z'_\mu \tag{10.54}$$

with algebraic rules given by

$$Z_i\,W\;\; W'\,Z'_j = \frac{Z_i\,Z'_j}{Q(Z,Z')}\,, \quad W\,f(Z,Z') = f(\sigma(Z),\sigma^{-1}(Z'))\,W\,, \tag{10.55}$$

where σ is the translation by $-\eta$ as in Theorem 9.5 and

$$Q(Z,Z') = J_{23}\,(Z_0\,Z'_0 + Z_1\,Z'_1) + Z_2\,Z'_2 - Z_3\,Z'_3\,.$$

Proof. The first equality follows from lemma 10.13 and the definition 10.50 of Z, Z', W, W'. The first algebraic rule follows from (10.51) and lemma 10.14.
To obtain the second we need to understand the transformation

$$\epsilon\;\psi(u+im/2) \to \epsilon\;\psi(u-\eta+im/2)\,,$$

and to compare it with σ^{-1} where σ is the translation by $-\eta$ as in Theorem 9.5.
By construction σ is the product (9.45) of two involutions $\sigma = I\circ I_0$ where I_0 just alters the sign of Z_0 (*cf.* (9.46)). Thus $\sigma^{-1} = I_0\circ I = I_0\circ\sigma\circ I_0$ and to show that the above tranformation is σ^{-1} it is enough to show that σ commutes with $I_0\circ I_3$ where $I_3(Z) = \epsilon\,Z$. This follows from the commutation of translations on the elliptic curve and can be checked directly using (9.47). □

11. Algebraic geometry and C^*-algebras

In this section we shall develop the basic relation between noncommutative differential geometry in the sense of [8] and noncommutative algebraic geometry. This will be obtained by abstracting the results of proposition 10.15 of subsection 10.6 and giving a general construction, independent of ϑ-functions, of a homomorphism from a quadratic algebra to a crossed product algebra constructed from the geometric data.

11.1. **Central Quadratic Forms and Generalised Cross-Products.**

Let $\mathcal{A} = A(V,R) = T(V)/(R)$ be a quadratic algebra. Its geometric data $\{E\,,\,\sigma\,,\,\mathcal{L}\}$ is defined in such a way that $\mathcal{A}$ maps homomorphically to a cross-product algebra obtained from sections of powers of the line bundle $\mathcal{L}$ on powers of the correspondence σ ([3]).
We shall begin by a purely algebraic result which considerably refines the above homomorphism and lands in a richer cross-product. We use the notations of section 5 for general quadratic algebras.

Definition 11.1. *Let $Q \in S^2(V)$ be a symmetric bilinear form on V^* and C a component of $E \times E$. We shall say that Q is central on C iff for all $(Z,\,Z')$ in C and $\omega \in R$ one has,*

$$\omega(Z,Z')\,Q(\sigma(Z'),\sigma^{-1}(Z)) + Q(Z,Z')\,\omega(\sigma(Z'),\sigma^{-1}(Z)) = 0 \tag{11.1}$$

By construction the space of symmetric bilinear form on V^* which are central on C is a linear subspace of $S^2(V)$. Let C be a component of $E \times E$ globally invariant under the map

$$\tilde{\sigma}(Z,Z') :=\ (\sigma(Z),\sigma^{-1}(Z')) \tag{11.2}$$

Given a quadratic form Q central and not identically zero on the component C, we define as follows an algebra C_Q as a generalised cross-product of the ring $\mathcal{R}$ of meromorphic functions on C by the transformation $\tilde{\sigma}$. Let $L,\,L' \in V$ be such that $L(Z)\,L'(Z')$ does not vanish identically on C. We adjoin two generators W_L and $W'_{L'}$ which besides the usual cross-product rules,

$$W_L\,f = (f\circ\tilde{\sigma})\,W_L\,,\quad W'_{L'}\,f = (f\circ\tilde{\sigma}^{-1})\,W'_{L'}\,,\quad \forall f\in\mathcal{R} \tag{11.3}$$

fulfill the following relations,

$$W_L\,W'_{L'} := \pi(Z,Z')\,,\qquad W'_{L'}\,W_L := \pi(\sigma^{-1}(Z),\sigma(Z')) \tag{11.4}$$

where the function $\pi(Z,Z')$ is given by the ratio,

$$\pi(Z,Z') := \frac{L(Z)\,L'(Z')}{Q(Z,Z')} \tag{11.5}$$

The a priori dependence on $L,\,L'$ is eliminated by the rules,

$$W_{L_2} := \frac{L_2(Z)}{L_1(Z)}\,W_{L_1} \qquad W'_{L'_2} :=\ W'_{L'_1}\,\frac{L'_2(Z')}{L'_1(Z')} \tag{11.6}$$

which allow to adjoin all W_L and $W'_{L'}$ for L and L' not identically zero on the projections of C, without changing the algebra and provides an intrinsic definition of C_Q.
Our first result is

Lemma 11.2. *Let Q be central and not identically zero on the component C.*
(i) The following equality defines a homomorphism ρ: $\mathcal{A} \mapsto C_Q$

$$\sqrt{2}\,\rho(Y) := \frac{Y(Z)}{L(Z)}\,W_L + W'_{L'}\,\frac{Y(Z')}{L'(Z')}\,,\qquad \forall Y\in V \tag{11.7}$$

(ii) If $\sigma^4 \neq \mathbb{1}$, then $\rho(Q) = 1$ where Q is viewed as an element of $T(V)/(R)$.

Proof. (i) Formula (11.7) is independent of $L,\,L'$ using (11.6) and reduces (up to $\sqrt{2}$) to $W_Y + W'_Y$ when Y is non-trivial on the two projections of C. It is enough to check that the $\rho(Y) \in C_Q$ fulfill the quadratic relations $\omega \in R$. Let $\omega \in R$

$$\omega(Z,Z') = \sum \omega_{ij}\,Y_i(Z)\,Y_j(Z')$$

viewed as a bilinear form on V^*. One has

$$2\sum \omega_{ij}\,\rho(Y_i)\,\rho(Y_j) = \sum \omega_{ij}\,(W_{Y_i} + W'_{Y_i})(W_{Y_j} + W'_{Y_j}) = \sum \omega_{ij}\,Y_i(Z)\,Y_j(\sigma(Z)\,W^2 +$$
$$\sum \omega_{ij}\,\frac{Y_i(Z)\,Y_j(Z')}{Q(Z,Z')} + \sum \omega_{ij}\,\frac{Y_i(\sigma(Z'))\,Y_j(\sigma^{-1}(Z))}{Q(\sigma^{-1}(Z),\sigma(Z'))} + \sum \omega_{ij}\,W'^{\,2}\,Y_i(\sigma^{-1}(Z'))\,Y_j(Z')$$

where

$$W^2 = \frac{1}{L(Z)L(\sigma(Z))}W_L^2\,, \quad W'^{\,2} = W_{L'}'^{\,2}\frac{1}{L'(\sigma^{-1}(Z'))L'(Z')}\,.$$

The vanishing of the terms in W^2 and in W'^2 is automatic by construction of the correspondence σ *i.e.* the equality

$$\omega(Z,\sigma(Z)) = 0\,, \quad \forall Z \in E\,.$$

The sum of the middle terms is just

$$\frac{\omega(Z,Z')}{Q(Z,Z')} + \frac{\omega(\sigma(Z'),\sigma^{-1}(Z))}{Q(\sigma^{-1}(Z),\sigma(Z'))} = 0\,,$$

as follows from definition 11.1 and the symmetry of Q.

(ii) The above computation shows that $\rho(Q) = 1$ provided one can show that $Q(Z,\sigma(Z)) = 0$ and $Q(\sigma^{-1}(Z'),Z') = 0$ for all Z, Z' in the projections E, E' of C. We assume that $\sigma^4(Z)$ is not identically equal to Z on each connected component of E (resp. E') and use (11.1) with $Z' = \sigma(Z)$. The first term vanishes and we get

$$\omega(\sigma^2(Z),\sigma^{-1}(Z))\,Q(Z,\sigma(Z)) = 0\,, \quad \forall \omega \in R\,.$$

Thus if $Q(Z,\sigma(Z))$ does not vanish identically on a given connected component E_1 of E one gets that

$$\omega(\sigma^2(Z),\sigma^{-1}(Z)) = 0\,, \quad \forall Z \in E_1\,, \quad \omega \in R\,,$$

so that $\sigma^{-1}(Z) = \sigma^3(Z)$ for all $Z \in E_1$ which contradicts the hypothesis. ☐

Let $\mathcal{A}_{\mathbf{u}} = C_{\mathrm{alg}}(\mathbb{C}^4(\mathbf{u}))$ at generic $\mathbf{u}$, then by proposition 10.2 the center of $\mathcal{A}_{\mathbf{u}}$ contains the three linearly dependent quadratic elements

$$Q_m := J_{k\ell}\,(Y_0^2 + Y_m^2) + Y_k^2 - Y_\ell^2 \tag{11.8}$$

Proposition 11.3. *Let $\mathcal{A}_{\mathbf{u}} = C_{\mathrm{alg}}(\mathbb{C}^4(\mathbf{u}))$ at generic $\mathbf{u}$, then each Q_m is central on $F_{\mathbf{u}} \times F_{\mathbf{u}}$ ($\subset E_{\mathbf{u}} \times E_{\mathbf{u}}$).*

Proof. One uses (9.45) to check the algebraic identity. ☐

Together with lemma 11.2 this yields non trivial homomorphisms of $\mathcal{A}_{\mathbf{u}}$ whose unitarity will be analysed in the next section. Note that for a general quadratic algebra $\mathcal{A} = A(V,R) = T(V)/(R)$ and a quadratic form $Q \in S^2(V)$, such that $Q \in \mathrm{Center}(\mathcal{A})$, it does not automatically follow that Q is central on $E \times E$. For instance Proposition 11.3 no longer holds on $F_{\mathbf{u}} \times \{e_\nu\}$ where e_ν is any of the four points of $E_{\mathbf{u}}$ not in $F_{\mathbf{u}}$. In fact let us describe in some details what happens in the case of the θ-deformations *i.e.* $C_+ = \{(\varphi,\varphi,0)\}$ (case 7). We take the notations of subsection 5.9 to write the characteristic variety as the union of six lines ℓ_j.

Proposition 11.4. *Let $C_{\mathrm{alg}}(\mathbb{R}^4_\varphi)$ for $\varphi \in C_+$, and Q_k be defined by* (10.1) *and* (10.5)*.*

1. *Each Q_k is central on $\ell_i \times \ell_j$ provided i and j belong to the same subsets $I = \{1,2\}$ and $J = \{3,4,5,6\}$.*
2. *The bilinear form Q_1 does not vanish identically on $\ell_i \times \ell_j$ iff $i = j$ for $i,j \in I$ and iff $i \neq j$ for $i,j \in J$.*

(3) *The bilinear form Q_2 vanishes identically on $\ell_2 \times \ell_j$ and $\ell_j \times \ell_2$ for all j.*

This is proved by direct computations. Note that since Q_1 fails to be central on $\ell_1 \times \ell_3$ for instance, it was crucial to "localize" the notion of central quadratic form to components of the square $E \times E$ of the characteristic variety E. It is of course also crucial to check the non-vanishing of Q when applying lemma 11.2, and the component ℓ_2 does not work for Q_2 in that respect.
The precise table for the vanishing of the form Q_2 is the following where $\neq$ at (i,j) means that Q_2 does not vanish identically on $\ell_i \times \ell_j$,

$$
\begin{array}{cccccc}
\neq & 0 & \neq & \neq & \neq & \neq \\
0 & 0 & 0 & 0 & 0 & 0 \\
\neq & 0 & 0 & \neq & \neq & 0 \\
\neq & 0 & \neq & 0 & 0 & \neq \\
\neq & 0 & \neq & 0 & 0 & \neq \\
\neq & 0 & 0 & \neq & \neq & 0
\end{array}
$$

11.2. **Positive Central Quadratic Forms on Quadratic $*$-Algebras.**

The algebra $\mathcal{A}_{\mathbf{u}}$, $\mathbf{u} \in T$ is by construction a *quadratic $*$-algebra* i.e. a quadratic complex algebra $\mathcal{A} = A(V, R)$ which is also a $*$-algebra with involution $x \mapsto x^*$ preserving the subspace V of generators. Equivalently one can take the generators of $\mathcal{A}$ (spanning V) to be hermitian elements of $\mathcal{A}$. In such a case the complex finite-dimensional vector space V has a real structure given by the antilinear involution $v \mapsto j(v)$ obtained by restriction of $x \mapsto x^*$. Since one has $(xy)^* = y^* x^*$ for $x, y \in \mathcal{A}$, it follows that the set R of relations satisfies

$$(j \otimes j)(R) = t(R) \tag{11.9}$$

in $V \otimes V$ where $t : V \otimes V \to V \otimes V$ is the transposition $v \otimes w \mapsto t(v \otimes w) = w \otimes v$. This implies

Lemma 11.5. *The characteristic variety is stable under the involution $Z \mapsto j(Z)$ and one has*

$$\sigma(j(Z)) = j(\sigma^{-1}(Z))$$

We let C be as above an invariant component of $E \times E$ we say that C is *j-real* when it is globally invariant under the involution

$$\tilde{j}(Z, Z') := (j(Z'), j(Z)) \tag{11.10}$$

By lemma 11.5 this involution commutes with the automorphism $\tilde{\sigma}$ (11.2) and one has

Proposition 11.6. *Let C be a j-real invariant component of $E \times E$ and Q central on C be such that*

$$\overline{Q(\tilde{j}(Z, Z'))} = Q(Z, Z'), \quad \forall (Z, Z') \in C, \tag{11.11}$$

(1) *The following turns the cross-product C_Q into a $*$-algebra,*

$$f^*(Z, Z') := \overline{f(\tilde{j}(Z, Z'))}, \qquad (W_L)^* = W'_{j(L)}, \qquad (W'_{L'})^* = W_{j(L')} \tag{11.12}$$

(2) *The homomorphism ρ of lemma 11.2 is a $*$-homomorphism.*

Proof. We used the transpose of j to define $j(L)$ in (11.12) by

$$j(L)(Z) = \overline{L(j(Z))}\,, \qquad \forall Z \in V^*. \tag{11.13}$$

The compatibility of the involution with (11.3) follows from the commutation of $\tilde{j}$ with $\tilde{\sigma}$.
Its compatibility with (11.4) follows from

$$\pi^*(Z,Z') := \Big(\frac{L(j(Z'))\,L'(j(Z))}{Q(\tilde{j}(Z,\,Z'))}\Big)^- = \frac{j(L')(Z)\,j(L)(Z')}{Q(Z,\,Z')}\,.$$

To check 2) one writes for $Y \in V$,

$$\rho(Y)^* = (W_Y + W'_Y)^* = W'_{j(Y)} + W_{j(Y)} = \rho(j(Y)\,.$$

□

We have treated so far Z and Z' as independent variables. We shall now restrict the above construction to the graph of j *i.e.* to $\{(Z,Z') \in C \,|\, Z' = j(Z)\}$. Composing ρ with the restriction to the subset $K = \{Z \,|\, (Z, j(Z)) \in C\}$ one obtains in fact a $*$-homomorphism θ of $\mathcal{A} = A(V,R)$ to a twisted cross-product C^*-algebra, $C(K) \times_{\sigma,\,\mathcal{L}} \mathbb{Z}$ which involves the full geometric data $(E,\sigma,\mathcal{L})$ and encodes the central quadratic form Q as a Hermitian metric on $\mathcal{L}$ provided Q fulfills the following *positivity*.

Definition 11.7. *Let C be a j-real invariant component of $E \times E$ and Q central on C. Then Q is positive on C iff it fullfills* (11.11) *and*

$$Q(Z, j(Z)) > 0\,, \quad \forall Z \in K\,.$$

One can then endow the line bundle $\mathcal{L}$ dual of the tautological line bundle on $P(V^*)$ with the Hermitian metric defined by

$$\langle f\,L,\, g\,L'\rangle_Q(Z) = f(Z)\,\overline{g(Z)}\,\frac{L(Z)\,\overline{L'(Z)}}{Q(Z,\,j(Z))} \qquad L, L' \in V, \quad Z \in K\ , \quad \forall f, g \in C(K)\,. \tag{11.14}$$

We view $f\,L$ and $g\,L'$ as sections of $\mathcal{L}$ and the right hand side of the formula as a function on K which expresses their inner product $\langle f\,L, g\,L'\rangle$. This defines a Hermitian metric on the restriction of $\mathcal{L}$ to K.

Before we proceed we need to describe the general notion due to Pimsner [23] of twisted cross product. Given a compact space K, an homeomorphism σ of K and a hermitian line bundle $\mathcal{L}$ on K we define the C^*-algebra $C(K) \times_{\sigma,\,\mathcal{L}} \mathbb{Z}$ as the twisted cross-product of $C(K)$ by the Hilbert C^*-bimodule associated to $\mathcal{L}$ and σ ([2], [23]).
We let for each $n \geq 0$, $\mathcal{L}^{\sigma^n}$ be the hermitian line bundle pullback of $\mathcal{L}$ by σ^n and (cf. [3], [27])

$$\mathcal{L}_n := \mathcal{L} \otimes \mathcal{L}^{\sigma} \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}} \tag{11.15}$$

We first define a $*$-algebra as the linear span of the monomials

$$\xi\,W^n\,, \quad W^{*n}\,\eta^*\,, \quad \xi\,,\eta \in C(K,\mathcal{L}_n) \tag{11.16}$$

with product given as in ([3], [27]) for $(\xi_1\,W^{n_1})\,(\xi_2\,W^{n_2})$ so that

$$(\xi_1\,W^{n_1})\,(\xi_2\,W^{n_2}) := (\xi_1 \otimes (\xi_2 \circ \sigma^{n_1}))\,W^{n_1+n_2} \tag{11.17}$$

We use the hermitian structure of $\mathcal{L}_n$ to give meaning to the products $\eta^*\,\xi$ and $\xi\,\eta^*$ for $\xi\,,\eta \in C(K,\mathcal{L}_n)$. The product then extends uniquely to an associative product of $*$-algebra fulfilling the following additional rules

$$(W^{*k}\,\eta^*)\,(\xi\,W^k) := (\eta^*\,\xi) \circ \sigma^{-k}\,, \qquad (\xi\,W^k)\,(W^{*k}\,\eta^*) := \xi\,\eta^* \tag{11.18}$$

The C^*-norm of $C(K) \times_{\sigma, \mathcal{L}} \mathbb{Z}$ is defined as for ordinary cross-products and due to the amenability of the group $\mathbb{Z}$ there is no distinction between the reduced and maximal norms. The latter is obtained as the supremum of the norms in involutive representations in Hilbert space. The natural positive conditional expectation on the subalgebra $C(K)$ shows that the C^*-norm restricts to the usual sup norm on $C(K)$.
To lighten notations in the next statement we abreviate $j(Z)$ as $\bar{Z}$, but one should take care that in general the expression for $j(Z)$ can differ from $\bar{Z}$ for instance with the notations of subsection 10.6 one gets $j(Z) = \epsilon\, \bar{Z}$.

Theorem 11.8. *Let $K \subset E$ be a compact σ-invariant subset and Q be central and strictly positive on $\{(Z, \bar{Z});\, Z \in K\}$. Let $\mathcal{L}$ be the restriction to K of the dual of the tautological line bundle on $P(V^*)$ endowed with the hermitian metric $\langle\ ,\ \rangle_Q$.*
(i) The equality $\sqrt{2}\,\theta(Y) := Y\,W + W^*\,\bar{Y}^*$ yields a $*$-homomorphism

$$\theta : \mathcal{A} = A(V, R) \mapsto C(K) \times_{\sigma, \mathcal{L}} \mathbb{Z}$$

(ii) For any $Y \in V$ the C^*-norm of $\theta(Y)$ fulfills

$$\mathrm{Sup}_K \|Y\| \leq \sqrt{2} \|\,\theta(Y)\| \leq 2\, \mathrm{Sup}_K \|Y\|$$

(iii) If $\sigma^4 \neq \mathbb{1}$, then $\theta(Q) = 1$ where Q is viewed as an element of $T(V)/(R)$.

Proof. (i) The subset $\tilde{K} = \{(Z, j(Z));\, Z \in K\} \subset C$ is globally invariant under $\tilde{\sigma}$ by lemma 11.5. Moreover $\tilde{j}$ defined in (11.10) is the identity on $\tilde{K}$. Each $L \in V$ defines a section of $\mathcal{L}$ and hence an element $L\,W \in C(K) \times_{\sigma, \mathcal{L}} \mathbb{Z}$. The definition (11.14) of the hermitian structure of $\mathcal{L}$ then shows that the elements $L\,W$ and $W^*\, j(L')^*$ of $C(K) \times_{\sigma, \mathcal{L}} \mathbb{Z}$ fulfill the same algebraic rules (11.3), (11.4) as the W_L and $W'_{L'}$ while the involution of $C(K) \times_{\sigma, \mathcal{L}} \mathbb{Z}$ is the restriction of the involution of proposition 11.11. Thus the conclusion follows from lemma 11.2.
(ii) Since $(YW)(YW)^* = Y^*Y$ the C^*-norm of YW is $\mathrm{Sup}_K\|Y\|$. It follows that $\sqrt{2}\|\,\theta(Y)\| \leq 2\,\mathrm{Sup}_K\|Y\|$. For any complex number u of modulus one the map $\xi W^n \to u^n\, \xi W^n$ extends to a $*$-automorphism of $C(K) \times_{\sigma, \mathcal{L}} \mathbb{Z}$. It follows taking $u = i$ that $\|Y\,W - W^*\,\bar{Y}^*\| = \|Y\,W + W^*\,\bar{Y}^*\|$ and $\mathrm{Sup}_K\|Y\| \leq \sqrt{2}\|\,\theta(Y)\|$.
(iii) follows from lemma 11.2.

□

We shall now apply this general result to the algebras $C_{\mathrm{alg}}(\mathbb{R}^4_\varphi)$. We take the quadratic form

$$Q(X,\, X') := \sum X^\mu\, X'^\mu \tag{11.19}$$

in the x-coordinates, so that Q is the canonical central element defining the sphere S^3_φ by the equation $Q = 1$. Proposition 11.3 shows that in the generic case *i.e.* for $\varphi \in A \cup B$, the quadratic form Q is central on $F_\varphi \times F_\varphi$ with obvious notations. The positivity of Q is automatic since in the x-coordinates the involution j_φ coming from the involution of the quadratic $*$-algebra $C_{\mathrm{alg}}(\mathbb{R}^4_\varphi)$ is simply complex conjugation $j_\varphi(Z) = \bar{Z}$, so that $Q(X,\, j_\varphi(X)) > 0$ for $X \neq 0$. We thus get,

Corollary 11.9. *Let $K \subset F_\varphi$ be a compact σ-invariant subset. The homomorphism θ of Theorem 11.8 is a unital $*$-homomorphism from $C_{\mathrm{alg}}(S^3_\varphi)$ to the cross-product $C^\infty(K) \times_{\sigma, \mathcal{L}} \mathbb{Z}$.*

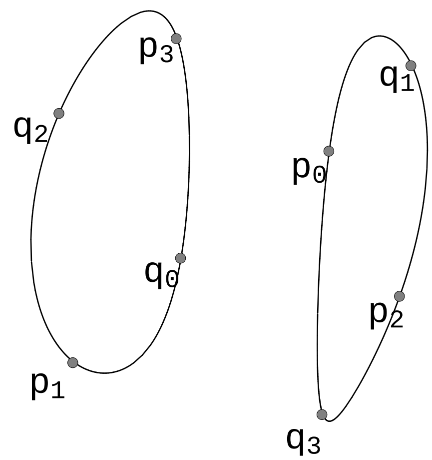

FIGURE 7. The Elliptic Curve $F_\varphi \cap P_3(\mathbb{R})$ (odd case)

This applies in particular to $K = F_\varphi$. It follows that one obtains a non-trivial C^*-algebra $C^*(S^3_\varphi)$ as the completion of $C_{\rm alg}(S^3_\varphi)$ for the semi-norm,

$$\|P\| := \operatorname{Sup}\| \pi(P)\| \tag{11.20}$$

where π varies through all unitary representations of $C_{\rm alg}(S^3_\varphi)$. It was clear from the start that (11.20) defines a finite C^*-semi-norm on $C_{\rm alg}(S^3_\varphi)$ since the equation of the sphere $\sum(x^\mu)^2 = 1$ together with the self-adjointness $x^\mu = x^{\mu *}$ show that in any unitary representation one has

$$\| \pi(x^\mu)\| \leq 1\,, \quad \forall \mu\,.$$

What the above corollary gives is a lower bound for the C^*-norm such as that given by statement (ii) of Theorem 11.8 on the linear subspace V of generators.

To analyse the compact σ-invariant subsets of F_φ for generic φ, we distinguish the *even* case which corresponds to all s_k having the same sign (*cf.* Figure 4) (and holds for instance for $\varphi \in A$) from the *odd* case when all s_k dont have the same sign. First note that in all cases the real curve $F_\varphi \cap P_3(\mathbb{R})$ is non empty (it contains p_0), and has two connected components since it is invariant under the Klein group H (9.32).

In the even case σ preserves each of the two connected components of the real curve $F_\varphi \cap P_3(\mathbb{R})$. In the odd case it permutes them (cf. Figure 7).

Proposition 11.10. *Let φ be generic and even.*

(i) Each connected component of $F_\varphi \cap P_3(\mathbb{R})$ is a minimal compact σ-invariant subset.

(ii) Let $K \subset F_\varphi$ be a compact σ-invariant subset, then K is the sum in the elliptic curve F_φ with origin p_0 of $K_{\mathbb{T}} = K \cap F_{\mathbb{T}}(\varphi)^0$ (cf. 9.35) with the component C_φ of $F_\varphi \cap P_3(\mathbb{R})$ containing p_0.

(iii) The cross-product $C(F_\varphi) \times_{\sigma, \mathcal{L}} \mathbb{Z}$ is isomorphic to the mapping torus of the automorphism β of the noncommutative torus $\mathbb{T}^2_\eta = C_\varphi \times_\sigma \mathbb{Z}$ acting on the generators by the matrix $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$.

Proof. (i) This holds if we assume that φ is "generic" so that the elliptic parameter η fullfills $\eta \notin \mathbb{Q}$. The diophantine approximations of η will play an important role later on.

(ii) By construction the abelian compact group F_φ is the product $\mathbb{T}_1 \times \mathbb{T}_2$ of the one-dimensional tori $\mathbb{T}_1 = C_\varphi$ and $\mathbb{T}_2 = F_{\mathbb{T}}(\varphi)^0$, *i.e.* the component of $F_{\mathbb{T}}(\varphi)$ containing q_0 (cf. 9.35). The translation σ is $\eta \times \mathrm{Id}$ and the action of η is minimal on $\mathbb{T}_1 = C_\varphi$.

(iii) The isomorphism class of the cross-product $C(F_\varphi) \times_{\sigma,\mathcal{L}} \mathbb{Z}$ depends of $\mathcal{L}$ only through its class as a hermitian line bundle on the two torus F_φ. This class is entirely specified by the first Chern class $c_1(\mathcal{L})$. By construction one gets $c_1(\mathcal{L}) = 4$ since the space of holomorphic sections of $\mathcal{L}$ is the 4-dimensional space V.

Let U_j be the generators of the algebra $C(\mathbb{T}^2_\eta)$ where the presentation of the algebra is

$$U_1\, U_2 = e^{2\pi i\eta}\, U_2\, U_1\,.$$

For any integer k let β_k be the automorphism of $C(\mathbb{T}^2_\eta)$ acting on the generators U_j by

$$\beta_k(U_1) := U_1\,, \qquad \beta_k(U_2) := U_1^k\, U_2\,. \tag{11.21}$$

By construction the mapping torus $T(\beta_k)$ of the automorphism β_k is given by the algebra $C(T(\beta_k))$ of continuous maps $s \in \mathbb{R} \mapsto x(s) \in C(\mathbb{T}^2_\eta)$ such that $x(s+1) = \beta_k(x(s))\,,\ \forall s \in \mathbb{R}$. We just need to show that $C(T(\beta_4))$ is isomorphic to $C(F_\varphi) \times_{\sigma,\mathcal{L}} \mathbb{Z}$ and this follows from the general isomorphism

$$C(T(\beta_k)) \simeq C(\mathbb{T}_1 \times \mathbb{T}_2) \times_{\eta\times\mathrm{Id}\,,\mathcal{L}} \mathbb{Z}\,, \tag{11.22}$$

(with $T_j = \mathbb{R}/\mathbb{Z}$) for any hermitian line bundle $\mathcal{L}$ on $\mathbb{T}_1 \times \mathbb{T}_2$ with $c_1(\mathcal{L}) = k$. To check this one chooses $\mathcal{L}$ so that its continuous sections $C(\mathbb{T}_1 \times \mathbb{T}_2, \mathcal{L})$ are scalar functions $f(u,m)$ with $u, m \in \mathbb{R}$ such that

$$f(u+1,m) = f(u,m)\,, \quad f(u,m+1) = e^{2\pi i k\, u}\, f(u,m)\,, \quad \forall u,\, m \in \mathbb{R}\,.$$

while its hermitian metric is given by

$$\langle f, g\rangle(u,m) = f(u,m)\,\overline{g(u,m)}\,, \quad \forall u,\, m \in \mathbb{R}\,.$$

One defines a map

$$\alpha\ :\quad C(T(\beta_k)) \to C(\mathbb{T}_1 \times \mathbb{T}_2) \times_{\eta\times\mathrm{Id},\,\mathcal{L}} \mathbb{Z}\,,$$

by writing for $x \in C(T(\beta_k))$, $x = (x(s))$, $x(s) \in C(\mathbb{T}^2_\eta)$ the Fourier expansion

$$x(s) = \sum x(s,n)\, U_2^n\,, \quad x(s,n) \in C(\mathbb{T}_1)\,.$$

Then the $x(s,n) \in C(\mathbb{T}_1)$ define sections

$$x_n \in C(\mathbb{T}_1 \times \mathbb{T}_2, \mathcal{L}_n)$$

and one just lets

$$\alpha(x) = \sum x_n\, W^n \in C(\mathbb{T}_1 \times \mathbb{T}_2) \times_{\eta\times\mathrm{Id},\,\mathcal{L}} \mathbb{Z}\,, \quad \forall x \in C(T(\beta_k))\,.$$

One then checks that this gives the required isomorphism (11.22). ☐

Corollary 11.11. *Let φ be generic and even, then $F_\varphi \times_{\sigma,\mathcal{L}} \mathbb{Z}$ is a noncommutative 3-manifold with an elliptic action of the three dimensional Heisenberg Lie algebra $\mathfrak{h}_3$ and an invariant trace τ.*

Proof. This follows [7] from proposition 11.10 (iii). One can construct directly the action of $\mathfrak{h}_3$ on $C^\infty(F_\varphi) \times_{\sigma,\mathcal{L}} \mathbb{Z}$ by choosing a constant (translation invariant) curvature connection ∇, compatible with the metric, on the hermitian line bundle $\mathcal{L}$ on F_φ (viewed in the C^∞-category not in the holomorphic

[7] It justifies the terminology "nilmanifold"

one). The two covariant differentials ∇_j corresponding to the two vector fields X_j on F_φ generating the translations of the elliptic curve, give a natural extension of X_j as the unique derivations δ_j of $C^\infty(F_\varphi) \times_{\sigma,\mathcal{L}} \mathbb{Z}$ fulfilling the rules,

$$\begin{aligned} \delta_j(f) &= X_j(f)\,, \quad \forall f \in C^\infty(F_\varphi) \\ \delta_j(\xi\, W) &= \nabla_j(\xi)\, W\,, \quad \forall \xi \in C^\infty(F_\varphi\,, \mathcal{L}) \end{aligned} \tag{11.23}$$

We let δ be the unique derivation of $C^\infty(F_\varphi) \times_{\sigma,\mathcal{L}} \mathbb{Z}$ corresponding to the grading by powers of W. It vanishes on $C^\infty(F_\varphi)$ and fulfills

$$\delta(\xi\, W^k) = i\, k\, \xi\, W^k \qquad \delta(W^{*k}\, \eta^*) = -i\, k\, W^{*k}\, \eta^* \tag{11.24}$$

Let $i\,\kappa$ be the constant curvature of the connection ∇, one gets

$$[\delta_1,\, \delta_2] = \kappa\, \delta\,, \quad [\delta,\, \delta_j] = 0 \tag{11.25}$$

which provides the required action of the Lie algebra $\mathfrak{h}_3$ on $C^\infty(F_\varphi) \times_{\sigma,\mathcal{L}} \mathbb{Z}$. □

It follows that one is exactly in the framework developped in [8]. We refer to [24] and [1] where these noncommutative manifolds were analysed in terms of crossed products by Hilbert C^*-bimodules.
Integration on the translation invariant volume form dv of F_φ gives the $\mathfrak{h}_3$-invariant trace τ,

$$\begin{aligned} \tau(f) &= \int f dv\,, \quad \forall f \in C^\infty(F_\varphi) \\ \tau(\xi\, W^k) &= \tau(W^{*k}\, \eta^*) \,=\, 0\,, \quad \forall k \neq 0 \end{aligned} \tag{11.26}$$

It follows in particular that the results of [8] apply to obtain the calculus. In particular the following gives the "fundamental class" as a 3-cyclic cocycle,

$$\tau_3(a_0,\, a_1,\, a_2,\, a_3) = \sum \epsilon_{ijk}\, \tau(a_0\, \delta_i(a_1)\, \delta_j(a_2)\, \delta_k(a_3)) \tag{11.27}$$

where the δ_j are the above derivations with $\delta_3 := \delta$.
We shall in fact describe the same calculus in greater generality in the last section which will be devoted to the computation of the Jacobian of the homomorphism θ of corollary 11.9.
Similar results hold in the odd case. Then $F_\varphi \cap P_3(\mathbb{R})$ is a minimal compact σ-invariant subset, any compact σ-invariant subset $K \subset F_\varphi$ is the sum in the elliptic curve F_φ with origin p_0 of $F_\varphi \cap P_3(\mathbb{R})$ with $K_{\mathbb{T}} = K \cap F_{\mathbb{T}}(\varphi)^0$ but the latter is automatically invariant under the subgroup $H_0 \subset H$ of order 2 of the Klein group H (9.32)

$$H_0 := \{h \in H \,|\, h(F_{\mathbb{T}}(\varphi)^0) = F_{\mathbb{T}}(\varphi)^0\} \tag{11.28}$$

The group law in F_φ is described geometrically as follows. It involves the point q_0. The sum $z = x+y$ of two points x and y of F_φ is $z = I_0(w)$ where w is the 4th point of intersection of F_φ with the plane determined by the three points $\{q_0, x, y\}$. It commutes by construction with complex conjugation so that $\overline{x+y} = \overline{x} + \bar{y}\,, \quad \forall x,\, y \in F_\varphi$.
By lemma 11.5 the translation σ is imaginary for the canonical involution j_φ. In terms of the coordinates Z_μ this involution is described as follows, using (9.36) (multiplied by $e^{i(\pi/4-\varphi_1-\varphi_2-\varphi_3)}2^{-3/2}$) to change variables. Among the 3 real numbers

$$\lambda_k \;=\; \cos\varphi_\ell \cos\varphi_m \sin(\varphi_\ell - \varphi_m)\,, \qquad k \in \{1,2,3\}$$

two have the same sign ϵ and one, λ_k, $k \in \{1,2,3\}$, the opposite sign. Then

$$j_\varphi = \epsilon\, I_k \,\circ\, c \tag{11.29}$$

where c is complex conjugation on the real elliptic curve F_φ (section 3) and I_μ the involution

$$I_\mu(Z_\mu) = -Z_\mu,\ I_\mu(Z_\nu) = Z_\nu\,, \quad \nu \neq \mu \tag{11.30}$$

The index k and the sign ϵ remain constant when φ varies in each of the four components of the complement of the four points q_μ in $F_{\mathbb{T}}(\varphi)$. The sign ϵ matters for the action of j_φ on linear forms as in (11.13), but is irrelevant for the action on F_φ. Each involution I_μ is a symmetry $z \mapsto p - z$ in the elliptic curve F_φ and the products $I_\mu \circ I_\nu$ form the Klein subgroup H (9.32) acting by translations of order two on F_φ.

The quadratic form Q of (11.19) is given in the new coordinates by,

$$Q = \left(\prod \cos^2\varphi_\ell\right) \sum t_k\, s_k\, Q_k \tag{11.31}$$

with $s_k := 1 + t_\ell\, t_m$, $t_k := \tan\varphi_k$ and Q_k defined by (10.8).

Let us assume that $0 < \varphi_1 < \varphi_2 < \varphi_3 < \pi/2$ for instance, then the relation between the x^μ and the Y_μ is given with the appropriate normalization of the Y_μ by $x^\mu = \rho_\mu\, Y_\mu$ where

$$\begin{aligned} \rho_0^2 &= -\sin(\varphi_1 - \varphi_2)\sin(\varphi_1 - \varphi_3)\sin(\varphi_2 - \varphi_3)\,, \quad \rho_1^2 = -\cos\varphi_2\cos\varphi_3\sin(\varphi_2 - \varphi_3)\,, \\ \rho_2^2 &= \cos\varphi_1\cos\varphi_3\sin(\varphi_1 - \varphi_3)\,, \quad \rho_3^2 = -\cos\varphi_1\cos\varphi_2\sin(\varphi_1 - \varphi_2))\,. \end{aligned} \tag{11.32}$$

All the ρ_μ are real except for ρ_2 which is purely imaginary and the involution j is $I_2 \circ c$. One checks directly that

$$\sum \rho_\mu^2\, Y_\mu^2 = \left(\prod \cos^2\varphi_\ell\right) \sum t_k\, s_k\, Q_k\,.$$

We can now compare the $*$-homomorphism $\tilde\rho$ of section 10.6 with the $*$-homomorphism obtained from a positive central quadratic form, one gets with the constants s and b given by,

$$s = -s_2 \prod \sin\varphi_j\,, \quad b^2 = \prod \cos\varphi_j \cos(\varphi_k - \varphi_\ell)\,.$$

Proposition 11.12. *Let* $0 < \varphi_1 < \varphi_2 < \varphi_3 < \pi/2$.
(i) The $*$-homomorphism $\tilde\rho$ is the $*$-homomorphism associated to the central quadratic form Q',

$$s\, Q' = Q_1 + Q_3 + s_2\, Q_2\,,$$

which is positive on E for the involution $I_3 \circ c$.
(ii) Let

$$\beta(Y_0) = i\, Y_2\,, \quad \beta(Y_1) = \sqrt{J_{12}}\, Y_3\,, \quad \beta(Y_2) = -\sqrt{J_{12}}\sqrt{J_{23}}\, Y_0\,, \quad \beta(Y_3) = i\sqrt{J_{23}}\, Y_1\,,$$

the map $b\,\beta$ gives a $*$-isomorphism sending the form Q' into Q and the involution $I_3 \circ c$ into $I_2 \circ c$.

Proof. (i) By proposition 10.15 it is enough to show that Q' corresponds to $\sum (x^\mu)^2$ under the composition of the transformations $S_\mu = \lambda_\mu\, x^\mu$ of lemma 4.4 and (10.48)

$$S_0 = d\, Y_2\,, \quad S_1 = i\, Y_3\,, \quad S_2 = d\, Y_0\,, \quad S_3 = -Y_1\,,$$

where $d^2 = -J_{31}$.
(ii) The map $b\,\beta$ is obtained as the composition of the isomorphisms (10.48), $S_\mu = \lambda_\mu\, x^\mu$ of lemma 4.4 and $x^\mu = \rho_\mu\, Y_\mu$ with ρ_μ given in (11.32). Thus the answer follows since each of these maps is a $*$-isomorphism and the image of Q' in the x^μ variables is simply $\sum (x^\mu)^2$.

$\square$

Let φ be generic and even and $v \in F_{\mathbb{T}}(\varphi)^0$. Let $K(v) = v + C_\varphi$ be the minimal compact σ-invariant subset containing v (Proposition 11.10 (ii)). By Corollary 11.9 we get a homomorphism,

$$\theta_v \;:\; C_{\mathrm{alg}}(S^3_\varphi) \mapsto C^\infty(\mathbb{T}^2_\eta) \tag{11.33}$$

whose non-triviality will be proved below in corollary 12.8. We shall first show (Theorem 11.13) that it transits through the cross-product of the field K_q of meromorphic functions on the elliptic curve by the subgroup of its Galois group $\mathrm{Aut}_{\mathbb{C}}(K_q)$ generated by the translation σ.

For $Z = v + z$, $z \in C_\varphi$, one has using (11.29) and (9.35),

$$j_\varphi(Z) = I_\mu(Z - v) - I(v) \tag{11.34}$$

which is a rational function $r(v, Z)$. Fixing φ, v and substituting Z and $Z' = r(v, Z)$ in the formulas (11.4) and (11.5) of lemma 11.2 with L real such that $0 \notin L(K(v))$, $L' = \epsilon L \circ I_\mu$ and Q given by (11.31) we obtain rational formulas for a homomorphism $\tilde{\theta_v}$ of $C_{\mathrm{alg}}(S^3_\varphi)$ to the generalised cross-product of the field K_q of meromorphic functions $f(Z)$ on the elliptic curve F_φ by σ. The generalised cross-product rule (11.4) is given by $W\,W' := \gamma(Z)$ where γ is a rational function. Similarly $W'\,W := \gamma(\sigma^{-1}(Z))$. Using integration on the cycle $K(v)$ to obtain a trace, together with corollary 11.9, we get,

Theorem 11.13. *The homomorphism $\theta_v : C_{\mathrm{alg}}(S^3_\varphi) \mapsto C^\infty(\mathbb{T}^2_\eta)$ factorises with a homomorphism $\tilde{\theta_v} : C_{\mathrm{alg}}(S^3_\varphi) \mapsto K_q \times_\sigma \mathbb{Z}$ to the generalised cross-product of the field K_q of meromorphic functions on the elliptic curve F_φ by the subgroup of the Galois group $Aut_{\mathbb{C}}(K_q)$ generated by σ. Its image generates the hyperfinite factor of type II_1 after weak closure relative to the trace given by integration on the cycle $K(v)$.*

Elements of K_q with poles on $K(v)$ are unbounded and give elements of the regular ring of affiliated operators, but all elements of $\theta_v(C_{\mathrm{alg}}(S^3_\varphi))$ are regular on $K(v)$. The above generalisation of the cross-product rules (11.4) with the rational formula for $W\,W' := \gamma(Z)$ is similar to the introduction of 2-cocycles in the standard Brauer theory of central simple algebras.

12. The Jacobian of the Covering of S^3_φ

In this section we shall analyse the morphism of $*$-algebras

$$\theta \;:\; C_{\mathrm{alg}}(S^3_\varphi) \mapsto C^\infty(F_\varphi \times_{\sigma,\mathcal{L}} \mathbb{Z}) \tag{12.1}$$

of Corollary 11.9, by computing its Jacobian in the sense of noncommutative differential geometry ([9]). We postpone the analysis at the C^*-level, in particular the role of the discrete series, to another forthcoming publication.

The usual Jacobian of a smooth map $\varphi : M \mapsto N$ of manifolds is obtained as the ratio $\varphi^*(\omega_N)/\omega_M$ of the pullback of the volume form ω_N of the target manifold N with the volume form ω_M of the source manifold M. In noncommutative geometry, differential forms ω of degree d become Hochschild classes $\tilde{\omega} \in HH_d(\mathcal{A})$, $\mathcal{A} = C^\infty(M)$. Moreover since one works with the dual formulation in terms of algebras, the pullback $\varphi^*(\omega_N)$ is replaced by the pushforward $\varphi^t_*(\tilde{\omega}_N)$ under the corresponding transposed morphism of algebras $\varphi^t(f) := f \circ \varphi$, $\forall f \in C^\infty(N)$.

The noncommutative sphere S^3_φ admits a canonical "volume form" given by the Hochschild 3-cycle $\mathrm{ch}_{\frac{3}{2}}(U)$. Our goal is to compute the push-forward,

$$\theta_*(\mathrm{ch}_{\frac{3}{2}}(U)) \in HH_3(C^\infty(F_\varphi \times_{\sigma,\mathcal{L}} \mathbb{Z})) \tag{12.2}$$

Let φ be generic and even. The noncommutative manifold $F_\varphi \times_{\sigma,\mathcal{L}} \mathbb{Z}$ is, by Corollary 11.11, a noncommutative 3-dimensional nilmanifold isomorphic to the mapping torus of an automorphism of the noncommutative 2-torus T^2_η. Its Hochschild homology is easily computed using the corresponding result for the noncommutative torus ([9]). It admits in particular a canonical volume form $V \in HH_3(C^\infty(F_\varphi \times_{\sigma,\mathcal{L}} \mathbb{Z}))$ which corresponds to the natural class in $HH_2(C^\infty(T^2_\eta))$ ([9]). The volume form V is obtained in the cross-product $F_\varphi \times_{\sigma,\mathcal{L}} \mathbb{Z}$ from the translation invariant 2-form dv on F_φ.

To compare $\theta_*(\mathrm{ch}_{\frac{3}{2}}(U))$ with V we shall pair it with the 3-dimensional Hochschild cocycle $\tau_h \in HH^3(C^\infty(F_\varphi \times_{\sigma,\mathcal{L}} \mathbb{Z}))$ given, for any element h of the center of $C^\infty(F_\varphi \times_{\sigma,\mathcal{L}} \mathbb{Z})$, by

$$\tau_h(a_0, a_1, a_2, a_3) = \tau_3(h\, a_0, a_1, a_2, a_3) \tag{12.3}$$

where $\tau_3 \in HC^3(C^\infty(F_\varphi \times_{\sigma,\mathcal{L}} \mathbb{Z}))$ is the fundamental class in cyclic cohomology defined by (11.27).

By proposition 10.4 the component $\mathrm{ch}_{\frac{3}{2}}(U)$ of the Chern character is given by,

$$\begin{aligned} \mathrm{ch}_{\frac{3}{2}}(U) = \quad & \sum \epsilon_{\alpha\beta\gamma\delta} \cos(\varphi_\alpha - \varphi_\beta + \varphi_\gamma - \varphi_\delta)\ x^\alpha\, dx^\beta\, dx^\gamma\, dx^\delta\ - \\ i\ & \sum \sin 2(\varphi_\mu - \varphi_\nu)\ x^\mu\, dx^\nu\, dx^\mu\, dx^\nu \end{aligned} \tag{12.4}$$

where $\varphi_0 := 0$. In terms of the Y_μ one gets,

$$\begin{aligned} \mathrm{ch}_{\frac{3}{2}}(U) \quad = \quad & \kappa \sum \delta_{\alpha\beta\gamma\delta}\, (s_\alpha - s_\beta + s_\gamma - s_\delta)\ Y_\alpha\, dY_\beta\, dY_\gamma\, dY_\delta\ + \\ & \kappa \sum \epsilon_{\alpha\beta\gamma\delta}\ (s_\alpha\ - s_\beta)\ \ Y_\gamma\, dY_\delta\, dY_\gamma\, dY_\delta \end{aligned} \tag{12.5}$$

where $s_0 := 0$, $s_k := 1 + t_\ell\, t_m$, $t_k := \tan \varphi_k$ and

$$\delta_{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta\gamma\delta}\, (n_\alpha - n_\beta + n_\gamma - n_\delta) \tag{12.6}$$

with $n_0 = 0$ and $n_k = 1$. The normalization factor is

$$\kappa = i \prod \cos^2(\varphi_k) \sin(\varphi_\ell - \varphi_m) \tag{12.7}$$

Formula (12.5) shows that, up to normalization, $\mathrm{ch}_{\frac{3}{2}}(U)$ only depends on the fiber F_φ of φ.

Let φ be generic and even, there is a similar formula in the odd case. In our case the involutions I and I_0 are conjugate by a real translation κ of the elliptic curve F_φ and we let $F_\varphi(0)$ be one of the two connected components of,

$$\{Z \in F_\varphi \mid I_0(Z) = \bar{Z}\} \tag{12.8}$$

By Proposition 11.10 we can identify the center of $C^\infty(F_\varphi \times_{\sigma,\mathcal{L}} \mathbb{Z})$ with $C^\infty(F_\varphi(0))$. We assume for simplicity that $\varphi_j \in [0, \frac{\pi}{2}]$ are in cyclic order $\varphi_k < \varphi_l < \varphi_m$ for some $k \in \{1, 2, 3\}$.

Theorem 12.1. *Let $h \in$ Center $(C^\infty(F_\varphi \times_{\sigma,\mathcal{L}} \mathbb{Z})) \sim C^\infty(F_\varphi(0))$. Then*

$$\langle ch_{\frac{3}{2}}(U), \tau_h \rangle = 6\,\pi\,\Omega \int_{F_\varphi(0)} h(Z)\, dR(Z)$$

where Ω is the period given by the elliptic integral of the first kind,

$$\Omega = \int_{C_\varphi} \frac{Z_k dZ_0 - Z_0 dZ_k}{Z_\ell Z_m}$$

and R the rational fraction,

$$R(Z) = t_k\, \frac{Z_m^2}{Z_m^2 + c_k\, Z_l^2}$$

with $c_k = \mathrm{tg}(\varphi_l) \cot(\varphi_k - \varphi_\ell)$.

We assume that $0 < \varphi_1 < \varphi_2 < \varphi_3 < \pi/2$ *i.e.* that $k = 1$. We start from corollary 10.11 and express the result

$$\omega = -\frac{\sigma(m)^4\, g(m)}{\lambda\,\Lambda}\, dm\,, \tag{12.9}$$

in terms of the trigonometric parameters φ_j and the coordinates Z_μ of Z.
One has (*cf.* (10.36))

$$\sigma(m)^4 = (\prod \sin\varphi_j)^2\, (C_1 - \lambda\, C_2)^{-2}\,, \tag{12.10}$$

and we begin by giving a better formula for $C_1 - \lambda\, C_2$.

Lemma 12.2. *One has*

$$C_1 - \lambda\, C_2 = b_1\, \vartheta_1^2(im) + b_2\, \vartheta_2^2(im)\,, \tag{12.11}$$

where

$$b_1 = 4\,\frac{\vartheta_1^2(\eta)}{s_1\, \vartheta_2^2(\eta)}\,, \quad b_2 = 4\,\frac{s_1\, - 1}{s_1}\,. \tag{12.12}$$

Proof. One uses (10.37), $\lambda = \dfrac{\vartheta_2^2(0)}{s_1\, \vartheta_2^2(\eta)}$ and the identity

$$\vartheta_2^2(0)\, \vartheta_2(\eta + im)\, \vartheta_2(\eta - im) = \vartheta_2^2(im)\, \vartheta_2^2(\eta) - \vartheta_1^2(im)\, \vartheta_1^2(\eta)\,.$$

∎

The m-dependent terms are understood from the following lemma

Lemma 12.3. *Let b_j be arbitrary constants, then*

$$\vartheta_3^2(0)\vartheta_4^2(0)\, \frac{d}{du}\, \frac{\vartheta_1^2(u)}{b_1\, \vartheta_1^2(u) + b_2\, \vartheta_2^2(u)} = b_2\, \frac{\vartheta_1'(0)^3}{\pi^2}\, \frac{\vartheta_1(2u)}{(b_1\, \vartheta_1^2(u) + b_2\, \vartheta_2^2(u))^2}\,. \tag{12.13}$$

Proof. This follows from the classical identity

$$\frac{d}{du}\, \mathrm{sn}(u) = \mathrm{cn}(u)\, \mathrm{dn}(u)\,, \tag{12.14}$$

which implies

$$\frac{d}{du}\, \frac{\vartheta_1^2(u)}{\vartheta_2^2(u)} = 2\,\pi\, \vartheta_2^2(0)\, \frac{\vartheta_1(u)\vartheta_2(u)\vartheta_3(u)\vartheta_4(u)}{\vartheta_2^4(u)} = \frac{\vartheta_1'(0)^3}{\pi^2\, \vartheta_3^2(0)\vartheta_4^2(0)}\, \frac{\vartheta_1(2u)}{\vartheta_2^4(u)}\,,$$

using the duplication formula

$$\vartheta_2(0)\vartheta_3(0)\vartheta_4(0)\vartheta_1(2\,u) = 2\, \vartheta_1(u)\vartheta_2(u)\vartheta_3(u)\vartheta_4(u)$$

and the Jacobi derivative formula

$$\frac{\vartheta_1'(0)}{\pi} = \vartheta_2(0)\vartheta_3(0)\vartheta_4(0)$$

∎

Taking $u = im$ in (12.13) this allows to write ω as the differential of

$$R(m) = \frac{c\, \vartheta_1^2(im)}{b_1\, \vartheta_1^2(im) + b_2\, \vartheta_2^2(im)} \tag{12.15}$$

using (10.45)

$$g(m) = 24\,(2\pi i)^3\, \frac{\vartheta_1'(0)^3}{\pi^3}\, \frac{\vartheta_1(\eta)\, \vartheta_1(2im)}{\vartheta_2(\eta)\vartheta_3(\eta)\vartheta_4(\eta)}\,. \tag{12.16}$$

The constant c is uniquely determined and will be simplified

later, we get so far,

$$c = 24\,(2\pi)^3\,\frac{(\prod \sin\varphi_j)^2\,\vartheta_1(\eta)\,\vartheta_3^2(0)\vartheta_4^2(0)}{\pi\, b_2\,\lambda\,\Lambda\,\vartheta_2(\eta)\vartheta_3(\eta)\vartheta_4(\eta)}\,. \tag{12.17}$$

Lemma 12.4. *(i) The differential form*

$$\chi := \frac{Z_k dZ_0 - Z_0 dZ_k}{s_k\, Z_\ell Z_m} \tag{12.18}$$

is independent of k and is, up to scale, the only holomorphic form of type $(1,0)$ on F_φ.
(ii) One has

$$2\,\pi\,\vartheta_4^2(0) = \frac{\vartheta_1(\eta)\,\vartheta_4(\eta)}{\vartheta_2(\eta)\,\vartheta_3(\eta)}\int_{C_\varphi}\frac{Z_3 dZ_0 - Z_0 dZ_3}{Z_1 Z_2} \tag{12.19}$$

Proof. (i) Recall that the equations defining F_φ are

$$\frac{Z_0^2 - Z_1^2}{s_1} = \frac{Z_0^2 - Z_2^2}{s_2} = \frac{Z_0^2 - Z_3^2}{s_3}\,. \tag{12.20}$$

One gets the required independence by differentiation.
(ii) Let us check (12.19). The factor $\frac{\vartheta_1(\eta)\,\vartheta_4(\eta)}{\vartheta_2(\eta)\,\vartheta_3(\eta)}$ allows to replace the Z_j by $\vartheta_{j+1}(2z)$ so that the right hand side gives using (12.14)

$$\int_0^1 \frac{\vartheta_4(2z)d\vartheta_1(2z) - \vartheta_1(2z)d\vartheta_4(2z)}{\vartheta_2(2z)\vartheta_3(2z)} = 2\,\pi\,\vartheta_4^2(0)\,.$$

□

In terms of $Z = (Z_j)$ one has

$$R(m) = c\, s_1\,\frac{\vartheta_1^2(\eta)}{4\,\vartheta_2^2(\eta)}\,\frac{Z_0^2}{a_1\, Z_0^2 + a_2\, Z_1^2} \tag{12.21}$$

where

$$a_1 = \frac{\vartheta_1^4(\eta)}{\vartheta_2^4(\eta)}\,, \quad a_2 = s_1 - 1\,. \tag{12.22}$$

One can express the coefficient a_1 in trigonometric terms using

Lemma 12.5.

$$a_1 = \frac{\vartheta_1^4(\eta)}{\vartheta_2^4(\eta)} = \frac{(s_1 - s_2)(s_1 - s_3)}{s_2 s_3} \tag{12.23}$$

Proof. By homogeneity of the right hand side one can replace the s_j by the σ_j of (10.44) One then gets

$$\frac{(s_1 - s_2)(s_1 - s_3)}{s_2 s_3} = -\,J_{12}\,J_{31}$$

and the result follows from (10.26). □

We now simplify the product $c\, s_1\,\frac{\vartheta_1^2(\eta)}{4\,\vartheta_2^2(\eta)}$ replacing $\vartheta_4^2(0)$ in (12.17) by (12.19) and using (*cf.* (10.12))

$$\Lambda = \prod_1^3 \left(\tan(\varphi_j)\,\cos(\varphi_k - \varphi_\ell)\right), \quad \lambda = \frac{\vartheta_3^2(0)}{s_2\,\vartheta_3^2(\eta)}\,. \tag{12.24}$$

By elementary computations and using once more (12.23) one gets

$$c\,s_1\,\frac{\vartheta_1^2(\eta)}{4\,\vartheta_2^2(\eta)} = 6\,\pi\,t_1\,a_1\,\frac{s_1}{s_3}\int_{C_\varphi}\frac{Z_3 dZ_0 - Z_0 dZ_3}{Z_1 Z_2} \tag{12.25}$$

We thus obtain so far using the elementary equality

$$\frac{a_2}{a_1} = \cot(\varphi_1 - \varphi_2)\cot(\varphi_1 - \varphi_3)$$

and lemma 12.4 which allows to eliminate the term $\frac{s_1}{s_3}$, using the definition of the period Ω the following formula for the rational fraction,

$$R(Z) = t_1\,\frac{Z_0^2}{Z_0^2 + a\,Z_1^2} \tag{12.26}$$

with $a = \cot(\varphi_1 - \varphi_2)\,\cot(\varphi_1 - \varphi_3)$.

What we have computed so far is the image of $\mathrm{ch}_{3/2}$ under the $*$-homomorphism $\tilde{\rho}$. By proposition 11.12 (i) this amounts to the image of $\mathrm{ch}_{3/2}$ under the $*$-homomorphism associated to the central quadratic form Q'. By proposition 11.12 (ii) we get the result for Q using the isomorphism β and this gives the following formula for $R(Z)$,

$$R(Z) = t_1\,\frac{Z_3^2}{Z_3^2 + c_1\,Z_2^2} \tag{12.27}$$

with $c_1 = \mathrm{tg}(\varphi_2)\,\cot(\varphi_1 - \varphi_2)$.
We shall explain below in section 13 how to perform the transition from (12.26) to (12.27) in a conceptual manner.

In fact the conceptual understanding of the simplicity of the final result of Theorem 12.1 is at the origin of many of the notions developped in the present paper and in particular of the "rational" formulation of the calculus which will be obtained in the last section. The geometric meaning of Theorem 12.1 is the computation of the Jacobian in the sense of noncommutative geometry of the morphism θ as explained above. The integral Ω is (up to a trivial normalization factor) a standard elliptic integral, it is given by an hypergeometric function in the variable

$$m := \frac{s_k(s_l - s_m)}{s_l(s_k - s_m)} \tag{12.28}$$

or a modular form in terms of q.

Lemma 12.6. *The differential of R is given on F_φ by $dR = J(Z)\,\chi$ where*

$$J(Z) = 2\,(s_l - s_m)\,c_k\,t_k\,\frac{Z_0\,Z_1\,Z_2\,Z_3}{(Z_m^2 + c_k\,Z_l^2)^2} \tag{12.29}$$

Proof. This can easily be checked using (12.14) but it is worthwile to give a simple direct argument. One has indeed by definition of χ

$$d\frac{Z_j^2}{Z_0^2} = -2\,s_j\,\frac{Z_0\,Z_1\,Z_2\,Z_3}{Z_0^4}\,\chi\,,$$

which gives using

$$s_l\,Z_m^2 - s_m\,Z_l^2 = (s_l - s_m)\,Z_0^2$$

the equality

$$d\frac{Z_l^2}{Z_m^2} = 2\,(s_m - s_l)\,\frac{Z_0\,Z_1\,Z_2\,Z_3}{Z_m^4}\,\chi\,,$$

and the required result. $\square$

The period Ω does not vanish and $J(Z)$, $Z \in F_\varphi(0)$, only vanishes on the 4 "ramification points" necessarily present due to the symmetries.

Corollary 12.7. *The Jacobian of the map θ^t is given by the equality*

$$\theta_*(ch_{\frac{3}{2}}(U)) = 3\,\Omega\,J\,V$$

where J is the element of the center $C^\infty(F_\varphi(0))$ of $C^\infty(F_\varphi \times_{\sigma,\mathcal{L}} \mathbb{Z})$ given by formula (12.29).

This statement assumes that φ is generic in the measure theoretic sense so that η admits good diophantine approximation ([9]). It justifies in particular the terminology of "ramified covering" applied to θ^t. The function J has only 4 zeros on $F_\varphi(0)$ which correspond to the ramification.

As shown by Theorem 9.5 the algebra $\mathcal{A}_\varphi$ is defined over $\mathbb{R}$, i.e. admits a natural antilinear automorphism of period two, γ uniquely defined by

$$\gamma(Y_\mu) := Y_\mu\,, \quad \forall \mu \tag{12.30}$$

Theorem 9.5 also shows that σ is defined over $\mathbb{R}$ and hence commutes with complex conjugation $c(Z) = \bar{Z}$. This gives a natural real structure γ on the algebra C_Q with $C = F_\varphi \times F_\varphi$ and Q as above,

$$\gamma(f(Z,Z')) := \overline{f(c(Z),c(Z'))}\,, \quad \gamma(W_L) := W_{c(L)}\,, \quad \gamma(W'_{L'}) := W'_{c(L')}$$

One checks that the morphism ρ of lemma 11.2 is "real" i.e. that,

$$\gamma \circ \rho = \rho \circ \gamma \tag{12.31}$$

Since $c(Z) = \bar{Z}$ reverses the orientation of F_φ, while γ preserves the orientation of S^3_φ it follows that $J(\bar{Z}) = -J(Z)$ and J necessarily vanishes on $F_\varphi(0) \cap P_3(\mathbb{R})$.

Note also that for general h one has $\langle \mathrm{ch}_{\frac{3}{2}}(U), \tau_h \rangle \neq 0$ which shows that both $\mathrm{ch}_{\frac{3}{2}}(U) \in HH_3$ and $\tau_h \in HH^3$ are non trivial Hochschild classes. These results hold in the smooth algebra $C^\infty(S^3_\varphi)$ containing the closure of $C_{\mathrm{alg}}(S^3_\varphi)$ under holomorphic functional calculus in the C^* algebra $C^*(S^3_\varphi)$. We can also use Theorem 12.1 to show the non-triviality of the morphism
$\theta_v : C_{\mathrm{alg}}(S^3_\varphi) \mapsto C^\infty(\mathbb{T}^2_\eta)$ of (11.33).

Corollary 12.8. *The pullback of the fundamental class $[\mathbb{T}^2_\eta]$ of the noncommutative torus by the homomorphism $\theta_v : C_{\mathrm{alg}}(S^3_\varphi) \mapsto C^\infty(\mathbb{T}^2_\eta)$ of (11.33) is non zero, $\theta_v^*([\mathbb{T}^2_\eta]) \neq 0 \in HH^2$ provided v is not a ramification point.*

We have shown above the non-triviality of the Hochschild homology and cohomology groups $HH_3(C^\infty(S^3_\varphi))$ and $HH^3(C^\infty(S^3_\varphi))$ by exhibiting specific elements with non-zero pairing. Combining the ramified cover $\pi = \theta^t$ with the natural spectral geometry (spectral triple) on the noncommutative 3-dimensional nilmanifold $F_\varphi \times_{\sigma,\mathcal{L}} \mathbb{Z}$ yields a natural spectral triple on S^3_φ in the generic case.

13. Calculus and Cyclic Cohomology

Theorem 12.1 suggests the existence of a "rational" form of the calculus explaining the appearance of the elliptic period Ω and the rationality of R. We shall show in this last section that this indeed the case. This will allow us to get a very simple conceptual form of the above computation of the Jacobian in Theorem 13.8 below.

Let us first go back to the general framework of twisted cross products of the form

$$\mathcal{A} = C^\infty(M) \times_{\sigma,\mathcal{L}} \mathbb{Z} \tag{13.1}$$

where σ is a diffeomorphism of the manifold M. We shall follow [10] to construct cyclic cohomology classes from cocycles in the bicomplex of group cohomology (with group $\Gamma = \mathbb{Z}$) with coefficients in de Rham currents on M. The twist by the line bundle $\mathcal{L}$ introduces a non-trivial interesting nuance. Let $\Omega(M)$ be the algebra of smooth differential forms on M, endowed with the action of $\mathbb{Z}$

$$\alpha_{1,k}(\omega) := \sigma^{*k}\omega\,, \quad k \in \mathbb{Z} \tag{13.2}$$

As in [12] p. 219 we let $\tilde{\Omega}(M)$ be the graded algebra obtained as the (graded) tensor product of $\Omega(M)$ by the exterior algebra $\wedge(\mathbb{C}[\mathbb{Z}]')$ on the augmentation ideal $\mathbb{C}[\mathbb{Z}]'$ in the group ring $\mathbb{C}[\mathbb{Z}]$. With $[n], n \in \mathbb{Z}$ the canonical basis of $\mathbb{C}[\mathbb{Z}]$, the augmentation $\epsilon : \mathbb{C}[\mathbb{Z}] \mapsto \mathbb{C}$ fulfills $\epsilon([n]) = 1, \forall n$, and

$$\delta_n := [n] - [0]\,, \quad n \in \mathbb{Z}\,, \quad n \neq 0 \tag{13.3}$$

is a linear basis of $\mathbb{C}[\mathbb{Z}]'$. The left regular representation of $\mathbb{Z}$ on $\mathbb{C}[\mathbb{Z}]$ restricts to $\mathbb{C}[\mathbb{Z}]'$ and is given on the above basis by

$$\alpha_{2,k}(\delta_n) := \delta_{n+k} - \delta_k\,, \quad k \in \mathbb{Z} \tag{13.4}$$

It extends to an action α_2 of $\mathbb{Z}$ by automorphisms of $\wedge\mathbb{C}[\mathbb{Z}]'$. We let $\alpha = \alpha_1 \otimes \alpha_2$ be the tensor product action of $\mathbb{Z}$ on the graded tensor product

$$\tilde{\Omega}(M) = \Omega(M) \otimes \wedge\mathbb{C}[\mathbb{Z}]'\,. \tag{13.5}$$

We now use the hermitian line bundle $\mathcal{L}$ to form the twisted cross-product

$$\mathcal{C} := \tilde{\Omega}(M) \times_{\alpha\,,\,\mathcal{L}} \mathbb{Z} \tag{13.6}$$

We let $\mathcal{L}_n$ be as in (11.15) for $n > 0$ and extend its definition for $n < 0$ so that $\mathcal{L}_{-n}$ is the pullback by σ^n of the dual $\hat{\mathcal{L}}_n$ of $\mathcal{L}_n$ for all n. The hermitian structure gives an antilinear isomorphism $* : \mathcal{L}_n \mapsto \hat{\mathcal{L}}_n$. One has for all $n, m \in \mathbb{Z}$ a canonical isomorphism

$$\mathcal{L}_n \otimes \sigma^{*m}\mathcal{L}_m \simeq \mathcal{L}_{n+m}\,, \tag{13.7}$$

which is by construction compatible with the hermitian structures.
The algebra $\mathcal{C}$ is the linear span of monomials $\xi\, W^n$ where

$$\xi \in C^\infty(M, \mathcal{L}_n) \otimes_{C^\infty(M)} \tilde{\Omega}(M) \tag{13.8}$$

with the product rules (11.17), (11.18).

Let ∇ be a hermitian connection on $\mathcal{L}$. We shall turn $\mathcal{C}$ into a differential graded algebra. By functoriality ∇ gives a hermitian connection on the $\mathcal{L}_k$ and hence a graded derivation

$$\nabla_n : C^\infty(M, \mathcal{L}_n) \otimes_{C^\infty(M)} \Omega(M) \mapsto C^\infty(M, \mathcal{L}_n) \otimes_{C^\infty(M)} \Omega(M) \tag{13.9}$$

whose square ∇_n^2 is multiplication by the curvature $\kappa_n \in \Omega^2(M)$ of $\mathcal{L}_n$,

$$\kappa_{n+m} = \kappa_n + \sigma^{*n}(\kappa_m)\,, \quad \forall n,\, m \in \mathbb{Z} \tag{13.10}$$

with $\kappa_1 = \kappa \in \Omega^2(M)$ the curvature of $\mathcal{L}$. Ones has $d\kappa_n = 0$ and we extend the differential d to a graded derivation on $\tilde{\Omega}(M)$ by

$$d\delta_n = \kappa_n \tag{13.11}$$

We can then extend ∇_n uniquely to the induced module

$$\mathcal{E}_n = C^\infty(M, \mathcal{L}_n) \otimes_{C^\infty(M)} \tilde{\Omega}(M) \tag{13.12}$$

by the equality

$$\tilde{\nabla}_n(\xi\,\omega) = \nabla_n(\xi)\,\omega + (-1)^{\deg(\xi)}\xi\, d\omega\,, \quad \forall \omega \in \tilde{\Omega}(M) \tag{13.13}$$

Proposition 13.1. *a) The pair $(\tilde{\Omega}(M),\, d)$ is a graded differential algebra.*
b) Let α be the tensor product action of $\mathbb{Z}$ then $\alpha(k) \in \mathrm{Aut}(\tilde{\Omega}(M),\, d)\,, \quad \forall k \in \mathbb{Z}$.
c) The following equality defines a flat connection on the induced module $\mathcal{E}_n$ on $\tilde{\Omega}(M)$,

$$\nabla'_n(\xi) = \tilde{\nabla}_n(\xi) - (-1)^{\deg(\xi)}\xi\,\delta_n\,. \tag{13.14}$$

d) The graded derivation d of $\tilde{\Omega}(M)$ extends uniquely to a graded derivation of $\mathcal{C}$ such that,

$$d(\xi\, W^n) = \nabla'_n(\xi)\, W^n \tag{13.15}$$

which turns the pair $(\mathcal{C},\, d)$ into a graded differential algebra.

Proof. a) By construction d is the unique extension of the differential d of $\Omega(M)$ to a graded derivation of the graded tensor product (13.5) such that (13.11) holds. One just needs to check that $d^2 = 0$ on simple tensors $\omega \otimes \delta_n$, one gets

$$d(\omega \otimes \delta_n) = \, d\omega \otimes \delta_n \,+\, (-1)^{\deg(\omega)}\,\omega\,\kappa_n \otimes 1\,,$$

$$d^2(\omega \otimes \delta_n) = \, d^2\omega \otimes \delta_n \,+\, (-1)^{\deg(\omega)+1}\, d\omega\,\kappa_n \otimes 1 \,+ (-1)^{\deg(\omega)}\, d\omega\,\kappa_n \otimes 1 \,= 0\,.$$

b) Let us check that $\alpha(k)$ commutes with the differentiation d on simple tensors $\omega \otimes \delta_n$. One has

$$d(\alpha(k)(\omega \otimes \delta_n)) = \, d(\sigma^{*k}(\omega) \otimes (\delta_{n+k} - \delta_k)) = \, \sigma^{*k}(d\omega) \otimes (\delta_{n+k} - \delta_k) + (-1)^{\deg(\omega)}\, \sigma^{*k}(\omega)\,(\kappa_{n+k} - \kappa_k) \otimes 1$$

$$\alpha(k)(d(\omega \otimes \delta_n)) = \, \sigma^{*k}(d\omega) \otimes (\delta_{n+k} - \delta_k) + (-1)^{\deg(\omega)}\, \sigma^{*k}(\omega\,\kappa_n) \otimes 1$$

and the equality follows from (13.10).
c) Since $\delta_n^2 = 0$ one gets

$$(\nabla'_n)^2(\xi) = \, (\tilde{\nabla}_n)^2(\xi) - \xi\, d\delta_n \,=\, 0$$

since $d\delta_n \,=\, \kappa_n$ is the curvature.
d) Since the algebra $\mathcal{C}$ is the linear span of monomials $\xi\, W^n$ the linear map d is well defined and coincides with the differential d on $\tilde{\Omega}(M)$ since $\delta_0 = 0$. Let us show that the two terms of (13.14) separately define derivations of $\mathcal{C}$. By construction the connections ∇_n are compatible with the canonical isomorphisms (13.7) and the same holds for their extensions $\tilde{\nabla}_n$ which is enough to show that the first term of (13.14) separately defines a derivation of $\mathcal{C}$. The proof for the second term

$$d^{'}(\xi\, W^n) = \, (-1)^{\deg(\xi)}\xi\,\delta_n\, W^n\,, \tag{13.16}$$

follows from (13.4) and is identical to the proof of lemma 12 chapter III of [12]. The flatness (c) of the connections ∇'_n ensures that $d^2 = 0$ so that $(\mathcal{C},\, d)$ is a graded differential algebra. $\square$

To construct closed graded traces on this differential graded algebra we follow ([10]) and consider the double complex of group cochains (with group $\Gamma = \mathbb{Z}$) with coefficients in de Rham currents on M. The cochains $\gamma \in C^{n,m}$ are totally antisymmetric maps from $\mathbb{Z}^{n+1}$ to the space $\Omega_{-m}(M)$ of de Rham currents of dimension $-m$, which fulfill

$$\gamma(k_0 + k, k_1 + k, k_2 + k, \cdots, k_n + k) = \, \sigma_*^{-k}\gamma(k_0, k_1, k_2, \cdots, k_n)\,, \quad \forall k,\, k_j \in \mathbb{Z} \tag{13.17}$$

Besides the coboundary d_1 of group cohomology, given by

$$(d_1\gamma)(k_0, k_1, \cdots, k_{n+1}) = \, \sum_0^{n+1} (-1)^{j+m}\, \gamma(k_0, k_1, \cdots, \hat{k_j}, \cdots, k_{n+1}) \tag{13.18}$$

and the coboundary d_2 of de Rham homology,

$$(d_2\gamma)(k_0, k_1, \cdots, k_n) = \, b(\gamma(k_0, k_1, \cdots, k_n))$$

the curvatures κ_n generate the further coboundary d_3 defined on Ker d_1 by,

$$(d_3\gamma)(k_0, \cdots, k_{n+1}) = \sum_0^{n+1} (-1)^{j+m+1} \kappa_{k_j} \gamma(k_0, \cdots, \hat{k_j}, \cdots, k_{n+1}) \tag{13.19}$$

which maps Ker $d_1 \cap C^{n,m}$ to $C^{n+1,m+2}$. Translation invariance follows from (13.10) and $\varphi_*(\omega C) = \varphi^{*-1}(\omega)\varphi_*(C)$ for $C \in \Omega_{-m}(M)$, $\omega \in \Omega^*(M)$.

To each $\gamma \in C^{n,m}$ one associates the functional $\tilde\gamma$ on $\mathcal{C}$ given by,

$$\begin{aligned} \tilde\gamma(\xi W^n) &= 0, \quad \forall n \neq 0, \quad \xi \in \tilde\Omega(M) \\ \tilde\gamma(\omega \otimes \delta_{k_1} \cdots \delta_{k_n}) &= \langle \omega, \gamma(0, k_1 \cdots, k_n)\rangle, \quad \forall k_j \in \mathbb{Z} \end{aligned} \tag{13.20}$$

One then has

Lemma 13.2. *Let $\gamma \in C^{n,m}$, then*

(i) One has for all $\rho \in \tilde\Omega(M)$,

$$\tilde\gamma(\rho - \alpha(-k)\rho) = -(\tilde{d_1\gamma})(\delta_k \rho). \tag{13.21}$$

(ii) One has for all $a, b \in \mathcal{C}$ with d' defined in (13.16)*,*

$$\tilde\gamma(a\, b - (-1)^{\deg(a)\deg(b)}\, b\, a) = (-1)^{\deg(a)} (\tilde{d_1\gamma})(a\, d'b). \tag{13.22}$$

(iii) One has for all $\rho \in \tilde\Omega(M)$,

$$\tilde\gamma(d\rho) = (\tilde{d_2\gamma})(\rho) + (\tilde{d_3\gamma})(\rho). \tag{13.23}$$

Proof. (i) We can assume that ρ is of the form

$$\rho = \omega \otimes \delta_{k_1} \cdots \delta_{k_n}$$

The left side of (13.21) is by construction

$$\langle \omega, \gamma(0, k_1 \cdots, k_n)\rangle - \tilde\gamma(\sigma^{*-k}(\omega) \otimes (\delta_{k_1-k} - \delta_{-k}) \cdots (\delta_{k_n-k} - \delta_{-k})).$$

When one expands the product one gets using $\delta_{-k}^2 = 0$,

$$(\delta_{k_1-k} - \delta_{-k}) \cdots (\delta_{k_n-k} - \delta_{-k}) = \prod \delta_{k_j-k} + \delta_{-k} \sum (-1)^i \prod_{j\neq i} \delta_{k_j-k}$$

and one uses the translation invariance (13.17) to write

$$\tilde\gamma(\sigma^{*-k}(\omega) \otimes \prod \delta_{k_j-k}) = \langle \omega, \gamma(k, k_1 \cdots, k_n)\rangle$$

and

$$\tilde\gamma(\sigma^{*-k}(\omega) \otimes (-1)^i \delta_{-k} \prod_{j\neq i} \delta_{k_j-k}) = -(-1)^i \langle \omega, \gamma(0, k, k_1 \cdots, \hat{k_i}, \cdots, k_n)\rangle$$

One thus obtains the same terms as in

$$-(\tilde{d_1\gamma})(\delta_k \rho) = -\langle \omega, (d_1\gamma)(0, k, k_1 \cdots, k_n)\rangle$$

using (13.18) and the graded commutation of δ_k with ω which yields a $(-1)^m$ overall sign.

(ii) It is enough to show that for any $k \in \mathbb{Z}$ and $a', b' \in \tilde\Omega(M)$ equation (11.13) holds for $a = a' W^k$ and $b = b' W^{-k}$. The graded commutativity of $\tilde\Omega(M)$ allows to write the graded commutator in (11.13) as $\rho - \alpha(-k)\rho$ where $\rho = a' \alpha(k)(b') \in \tilde\Omega(M)$. One has $\rho = a\, b$ and (i) thus shows that

$$\tilde\gamma(a\, b - (-1)^{\deg(a)\deg(b)}\, b\, a) = -(\tilde{d_1\gamma})(\delta_k\, a\, b) = -(-1)^{\deg(a)+\deg(b)} (\tilde{d_1\gamma})(a\, b\, \delta_k)$$

One has

$$a\,d^{'}(b) = \, a'\,\alpha(k)((-1)^{\deg(b)} b'\,\delta_{-k})$$

thus the result follows from the equality

$$\alpha(k)(\delta_{-k}) = \, -\,\delta_k\,.$$

(iii) We can assume that ρ is of the form

$$\rho = \, \omega \otimes \delta_{k_1} \cdots \delta_{k_n}$$

One has

$$d\rho = \, d\omega \otimes \delta_{k_1} \cdots \delta_{k_n} - \, (-1)^{\deg(\omega)} \sum (-1)^j \,\omega\, \kappa_{k_j} \otimes \delta_{k_1} \cdots \hat{\delta_{k_j}} \cdots \delta_{k_n}$$

which gives (13.23). □

To each $\gamma \in C^{n,m}$ one associates the $(n-m+1)$ linear form on $\mathcal{A} = C^\infty(M) \times_{\sigma,\,\mathcal{L}} \mathbb{Z}$ given by,

$$\Phi(\gamma)(a_0, a_1, \cdots, a_{n-m}) = \lambda_{n,m} \sum_0^{n-m} (-1)^{j(n-m-j)}\, \tilde{\gamma}(da_{j+1} \cdots da_{n-m}\, a_0\, da_1 \cdots da_{j-1}\, da_j) \tag{13.24}$$

where $\lambda_{n,m} := \frac{n!}{(n-m+1)!}$.

Lemma 13.3. *(i) The Hochschild coboundary $b\Phi(\gamma)$ is equal to $\Phi(d_1\gamma)$.*

(ii) Let $\gamma \in C^{n,m} \cap Ker\ d_1$. Then $\Phi(\gamma)$ is a Hochschild cocycle and

$$B\Phi(\gamma) = \Phi(d_2\gamma) + \tfrac{1}{n+1}\, \Phi(d_3\gamma)$$

Proof. (i) The proof is identical to that of Theorem 14 a) Chapter III of [12].

(ii) By (i) and the hypothesis $d_1(\gamma) = 0$ $\Phi(\gamma)$ is a Hochschild cocycle. In fact by lemma 13.2 the functional $\tilde{\gamma}$ is a graded trace and thus the formula for $\Phi(\gamma)$ simplifies to

$$\Phi(\gamma)(a_0, a_1, \cdots, a_{n-m}) = \, (n-m+1)\, \lambda_{n,m} \tilde{\gamma}\,(a_0\, da_1 \cdots da_{n-m}) \tag{13.25}$$

It follows that

$$B_0(\Phi(\gamma))(a_0, a_1, \cdots, a_{n-m-1}) = \, (n-m+1)\, \lambda_{n,m} \tilde{\gamma}\,(d\rho)\,, \quad \rho = \, a_0\, da_1 \cdots da_{n-m-1}\,,$$

and $B_0(\Phi(\gamma))$ is already cyclic so that $B(\Phi(\gamma)) = (n-m) B_0(\Phi(\gamma))$.

Since the coboundary d_2 anticommutes with d_1 one has $d_2\gamma \in \operatorname{Ker} d_1$ and $\Phi(d_2\gamma)$ is also a Hochschild cocycle and is given by (13.25) for $d_2\gamma$. Let us check that $d_3\gamma \in \operatorname{Ker} d_1$. One has up to an overall sign,

$$d_1\, d_3(\gamma) = \, \sum (-1)^i\, d_3(\gamma)(k_0, \cdots, \hat{k_i} \cdots, k_{n+2}) = \sum_{i,j} \epsilon(i,j)\, \kappa_{\,k_j} \gamma(k_0, \cdots, \hat{k_{i'}}, \cdots, \hat{k_{j'}}, \cdots, k_{n+1})$$

where (i', j') is the permutation of (i, j) such that $i' < j'$ and up to an overall sign $\epsilon(i, j)$ is the product of the signature of this permutation by $(-1)^{i+j}$. For each j the coefficient of $\kappa_{\,k_j}$ is up to an overall sign given by $d_1(\gamma)(k_0, \cdots, \hat{k_j} \cdots, k_{n+2}) = 0$, thus $d_1\, d_3(\gamma) = 0$.

The result thus follows from lemma 13.2 (iii). □

We shall now show how the above general framework allows to reformulate the calculus involved in Theorem 12.1 in rational terms. We let M be the elliptic curve F_φ where φ is generic and even. Let then ∇ be an arbitrary hermitian connection on $\mathcal{L}$ and κ its curvature. We first display a cocycle $\gamma = \sum \gamma_{n,m} \in \sum C^{n,m}$ which reproduces the cyclic cocycle τ_3.

Lemma 13.4. *There exists a two form α on $M = F_\varphi$ and a multiple $\lambda\, dv$ of the translation invariant two form dv such that :*

(i) $\kappa_n = n\,\lambda\, dv + (\sigma^{*n}\alpha - \alpha)\,, \quad \forall n \in \mathbb{Z}$

(ii) $d_2(\gamma_j) = 0\,, \quad d_1(\gamma_3) = 0\,, \quad d_1(\gamma_1) + \frac{1}{2}\, d_3(\gamma_3) = 0\,, \quad B\Phi(\gamma_1) = 0\,,$

where $\gamma_1 \in C^{1,0}$ and $\gamma_3 \in C^{1,-2}$ are given by

$\gamma_1(k_0, k_1) := \frac{1}{2}\,(k_1 - k_0)(\sigma^{*k_0}\alpha + \sigma^{*k_1}\alpha)\,, \quad \gamma_3(k_0, k_1) := k_1 - k_0\,, \quad \forall k_j \in \mathbb{Z}$

(iii) The class of the cyclic cocycle $\Phi(\gamma_1) + \Phi(\gamma_3)$ is equal to τ_3.

We use the generic hypothesis in the measure theoretic sense to solve the "small denominator" problem in (i). In (ii) we identify differential forms $\omega \in \Omega^d$ of degree d with the dual currents of dimension $2 - d$.

It is a general principle explained in [9] that a cyclic cocycle τ generates a calculus whose differential graded algebra is obtained as the quotient of the universal one by the radical of τ. We shall now explicitely describe the reduced calculus obtained from the cocycle of lemma 13.4 *(iii)*. We use as above the hermitian line bundle $\mathcal{L}$ to form the twisted cross-product

$$\mathcal{B} := \Omega(M) \times_{\alpha\,,\,\mathcal{L}} \mathbb{Z} \tag{13.26}$$

of the algebra $\Omega(M)$ of differential forms on M by the diffeomorphism σ. Instead of having to adjoin the infinite number of odd elements δ_n we just adjoin two χ and X as follows. We let δ be the derivation of $\mathcal{B}$ such that

$$\delta(\xi\, W^n) := i\, n\, \xi\, W^n\,, \quad \forall \xi \in C^\infty(M, \mathcal{L}_n) \otimes_{C^\infty(M)} \Omega(M) \tag{13.27}$$

We adjoin χ to $\mathcal{B}$ by tensoring $\mathcal{B}$ with the exterior algebra $\wedge\{\chi\}$ generated by an element χ of degree 1, and extend the connection ∇ (13.9) to the unique graded derivation d' of $\Omega' = \mathcal{B} \otimes \wedge\{\chi\}$ such that,

$$\begin{aligned} d'\,\omega &= \nabla\omega + \chi\delta(\omega)\,, \quad \forall \omega \in \mathcal{B} \\ d'\chi &= -\,\lambda\, dv \end{aligned} \tag{13.28}$$

with $\lambda\, dv$ as in lemma 13.4. By construction, every element of Ω' is of the form

$$y = b_0 + b_1\,\chi\,, \quad b_j \in \mathcal{B} \tag{13.29}$$

One does not yet have a graded differential algebra since $d'^2 \neq 0$. However, with α as in lemma 13.4 one has

$$d'^2(x) = [\,x,\, \alpha]\,, \quad \forall x \in \Omega' = \mathcal{B} \otimes \wedge\{\chi\} \tag{13.30}$$

and one can apply lemma 9 p.229 of [12] to get a differential graded algebra by adjoining the degree 1 element $X :=$ "$d1$" fulfilling the rules

$$X^2 = -\alpha\,, \quad x\, X\, y = 0\,, \quad \forall x, y \in \Omega' \tag{13.31}$$

and defining the differential d by,

$$\begin{aligned} d\,x &= d'\,x + [\,X,\, x]\,, \quad \forall x \in \Omega' \\ dX &= 0 \end{aligned} \tag{13.32}$$

where $[\,X,\,x]$ is the graded commutator. It follows from lemma 9 p.229 of [12] that we obtain a differential graded algebra Ω^*, generated by $\mathcal{B}$, ξ and X. In fact using (13.31) every element of Ω^* is of the form

$$x = x_{1,1} + x_{1,2}\,X + X\,x_{2,1} + X\,x_{2,2}X\,, \quad x_{i,j} \in \Omega' \tag{13.33}$$

and we define the functional $\int$ on Ω^* by extending the ordinary integral,

$$\int \omega := \int_M \omega\,, \quad \forall \omega \in \Omega(M) \tag{13.34}$$

first to $\mathcal{B} := \Omega(M) \times_{\alpha\,,\,\mathcal{L}} \mathbb{Z}$ by

$$\int \xi\, W^n \;:= 0\,, \quad \forall n \neq 0 \tag{13.35}$$

then to Ω' by

$$\int (b_0 + b_1\,\chi) := \int b_1\,, \quad \forall b_j \in \mathcal{B} \tag{13.36}$$

and finally to Ω^* as in lemma 9 p.229 of [12],

$$\int (x_{1,1} + x_{1,2}\,X + X\,x_{2,1} + X\,x_{2,2}X) := \int x_{1,1} + (-1)^{\deg(\,x_{2,2})} \int x_{2,2}\,\alpha \tag{13.37}$$

Theorem 13.5. *Let $M = F_\varphi$, ∇, α be as in lemma 13.4.*
The algebra Ω^* is a differential graded algebra containing $C^\infty(M) \times_{\alpha\,,\,\mathcal{L}} \mathbb{Z}$.
The functional $\int$ is a closed graded trace on Ω^*.
The character of the corresponding cycle on $C^\infty(M) \times_{\alpha\,,\,\mathcal{L}} \mathbb{Z}$

$$\tau(a_0, \cdots, a_3) := \int a_0\, da_1 \cdots da_3\,, \quad \forall a_j \in C^\infty(M) \times_{\alpha\,,\,\mathcal{L}} \mathbb{Z}$$

is cohomologous to the cyclic cocycle τ_3.

It is worth noticing that the above calculus fits with [8], [19], and [18].

Now in our case the line bundle $\mathcal{L}$ is holomorphic and we can apply Theorem 13.5 to its canonical hermitian connection ∇. We take the notations of section 11, with $C = F_\varphi \times F_\varphi$, and Q given by (11.31). This gives a particular "rational" form of the calculus which explains the rationality of the answer in Theorem 12.1. We first extend as follows the construction of C_Q. We let $\Omega(C, Q)$ be the generalised cross-product of the algebra $\Omega(C)$ of meromorphic differential forms (in dZ and dZ') on C by the transformation $\tilde{\sigma}$. The generators W_L and $W'_{L'}$ fulfill the cross-product rules,

$$W_L\;\omega = \tilde{\sigma}^*(\omega)\;W_L\,, \qquad W'_{L'}\,\omega = (\tilde{\sigma}^{-1})^*(\omega)\;W'_{L'} \tag{13.38}$$

while (11.4) is unchanged. The connection ∇ is the restriction to the subspace $\{Z' = \bar{Z}\}$ of the unique graded derivation ∇ on $\Omega(C, Q)$ which induces the usual differential on $\Omega(C)$ and satisfies,

$$\begin{aligned} \nabla W_L &= (\,d_Z\,\log L(Z) - d_Z\,\log Q(Z, Z'))\,W_L \\ \nabla W'_{L'} &= W'_{L'}\,(\,d_{Z'}\,\log L'(Z') - d_{Z'}\,\log Q(Z, Z')) \end{aligned} \tag{13.39}$$

where d_Z and $d_{Z'}$ are the (partial) differentials relative to the variables Z and Z'. Note that one needs to check that the involved differential forms such as $d_Z\,\log L(Z) - d_Z\,\log Q(Z, Z')$ are not only invariant under the scaling transformations $Z \mapsto \lambda Z$ but are also <u>basic</u>, i.e. have zero restriction to

the fibers of the map $\mathbb{C}^4 \mapsto P_3(\mathbb{C})$, in both variables Z and Z'. By definition the derivation $\delta_\kappa = \nabla^2$ of $\Omega(C, Q)$ vanishes on $\Omega(C)$ and fulfills

$$\delta_\kappa(W_L) = \kappa W_L \,, \qquad \delta_\kappa(W'_{L'}) = -W'_{L'}\,\kappa \tag{13.40}$$

where

$$\kappa = d_Z\, d_{Z'}\, \log Q(Z, Z') \tag{13.41}$$

is a basic form which when restricted to the subspace $\{Z' = \bar{Z}\}$ is the curvature. We let as above δ be the derivation of $\Omega(C, Q)$ which vanishes on $\Omega(C)$ and is such that $\delta W_L = i\, W_L$ and $\delta W'_{L'} = -i\, W'_{L'}$. We proceed exactly as above and get the graded algebras $\Omega' = \Omega(C, Q) \otimes \wedge\{\chi\}$ obtained by adjoining χ and Ω^* by adjoining X. We define d', d as in (13.28) and (13.32) and the integral $\int$ by integration (13.34) on the subspace $\{Z' = \bar{Z}\}$ followed as above by steps (13.35), (13.36), (13.37).

Corollary 13.6. *Let ρ: $C_{\rm alg}(S^3_\varphi) \mapsto C_Q$ be the morphism of lemma 11.2. The equality*

$$\tau_{\rm alg}(a_0, \cdots, a_3) := \int \rho(a_0)\, d^{'}\rho(a_1) \cdots d^{'}\rho(a_3)$$

defines a 3-dimensional Hochschild cocycle $\tau_{\rm alg}$ on $C_{\rm alg}(S^3_\varphi)$.
Let $h \in$ Center $(C^\infty(F_\varphi \times_{\sigma, \mathcal{L}} \mathbb{Z})) \sim C^\infty(F_\varphi(0))$. Then

$$\langle ch_{\frac{3}{2}}(U), \tau_h \rangle = \langle h\; ch_{\frac{3}{2}}(U), \tau_{\rm alg} \rangle$$

The computation of d' only involves rational fractions in the variables Z, Z' (13.39), and the formula (12.5) for $\mathrm{ch}_{\frac{3}{2}}(U)$ is polynomial in the W_L, $W'_{L'}$.
We are now ready for a better understanding and formulation of the result of Theorem 12.1. Indeed what the above shows is that the denominator that appears in the rational fraction $R(Z)$ of Theorem 12.1 should have to do with the central quadratic form $Q(Z, Z')$ evaluated on the pairs $(Z, \bar{Z})$. In fact the two dimensional space of central quadratic forms provides a natural space of functions of the form

$$R(Z) = \frac{P(Z, \bar{Z})}{Q(Z, \bar{Z})} \tag{13.42}$$

and this space is one dimensional when one mods out the constant functions.
The following lemma shows that these functions are in fact invariant under the correspondence σ.

Lemma 13.7. *Let Q be central and not identically zero on the component C and P be central then the function*

$$R(Z, Z') = \frac{P(Z, Z')}{Q(Z, Z')}$$

is invariant under $\tilde{\sigma}$.

It is thus natural now to compare the differential form dR with the form that appears in Theorem 12.1.

Theorem 13.8. *Let φ be generic and even and let Q be the central quadratic form on $\mathbb{R}^4_\varphi$ defining the three sphere S^3_φ. Let ρ_Q be the associated $*$-homomorphism*

$$C^\infty(S^3_\varphi) \to C^\infty(E \times_{\sigma, \mathcal{L}} \mathbb{Z})\,.$$

Then for any central quadratic form P not proportional to Q there exists a scalar μ such that

$$\langle ch_{\frac{3}{2}}(U), \tau_h \rangle = \mu \int h(Z)\, dR(Z)\,, \quad \forall h \in \text{Center}\; C^\infty(E \times_{\sigma, \mathcal{L}} \mathbb{Z})\,,$$

where

$$R(Z) = \frac{P(Z, \bar{Z})}{Q(Z, \bar{Z})}\,.$$

Proof. Let us show that one can interpret (12.26) in the above terms.
Thus with $a = \cot(\varphi_1 - \varphi_2)\cot(\varphi_1 - \varphi_3)$ we need to show that

$$R(Z) = t_1 \frac{Z_0^2}{Z_0^2 + a\, Z_1^2}$$

is in fact the restriction to the subset $F_\varphi(0) \subset E$ of elements Z with $c(Z) = I_0(Z)$ of a ratio of the form

$$\frac{P(Z, j(Z))}{Q(Z, j(Z))}\,.$$

One has $j(Z) = I_3(c(Z)$ and thus

$$j(Z) = I_3 \circ I_0(Z)\,, \quad \forall Z \in F_\varphi(0)\,. \tag{13.43}$$

We let Q' be the central quadratic form of proposition 11.12 namely

$$s\, Q' = Q_1 + Q_3 + s_2\, Q_2\,,$$

and we let P' be given by

$$s\, P' = Q_1 + Q_3\,, \tag{13.44}$$

A simple computation using (13.43) then shows that

$$b\, \frac{P'(Z, j(Z))}{Q'(Z, j(Z))} = \frac{Z_0^2}{Z_0^2 + a\, Z_1^2} - \frac{1}{s_1}\,, \quad \forall Z \in F_\varphi(0)\,, \tag{13.45}$$

where

$$b = \frac{\sin\varphi_2 \sin\varphi_3}{\cos(\varphi_2 - \varphi_3)}\,.$$

This gives the required result for the central quadratic form P' and since the space of central quadratic forms is two dimensional its quotient by multiples of Q' is one dimensional so that the result holds for all non-zero elements of this quotient. $\square$

With the above "invariant" formulation of the formulas of Theorem 12.1 we can now perform the change of variables required in the last part of its proof *i.e.* explain how to pass from (12.26) to (12.27).

We let P and Q be the central quadratic forms obtained from P' and Q' by the isomorphism β of proposition 11.12. The form Q is given by (11.31) *i.e.* by

$$\begin{aligned} Q \;&=\; (\prod \cos^2\varphi_\ell) \sum t_k\, s_k\, Q_k = \prod \sin(\varphi_\ell - \varphi_m)\, Z_0^2 \\ &- \sum \cos\varphi_\ell \cos\varphi_m \sin(\varphi_\ell - \varphi_m)\, Z_k^2 \end{aligned} \tag{13.46}$$

The form P is given by

$$P = \sum_1^3 \sin\varphi_k \sin(\varphi_\ell - \varphi_m) \cos(\varphi_k - \varphi_\ell - \varphi_m)\; Z_k^2 \tag{13.47}$$

The involution j_2 is now given by $j_2(Z) = I_2(c(Z)$ and one has

$$j_2(Z) = I_2 \circ I_0(Z)\,, \quad \forall Z \in F_\varphi(0)\,. \tag{13.48}$$

A simple computation using (13.48) then shows that

$$b\,\frac{P(Z, j_2(Z))}{Q(Z, j_2(Z))} = \frac{Z_3^2}{Z_3^2 + c_1\,Z_2^2} - \frac{1}{s_1}\,, \quad \forall Z \in F_\varphi(0)\,, \tag{13.49}$$

with $c_1 = \mathrm{tg}(\varphi_2)\cot(\varphi_1 - \varphi_2)$ and we thus obtain the formula required by Theorem 12.1.

14. Appendix 1: The list of minors

We give for convenience the list of the 15 minors of the matrix (5.3), with labels the missing lines, and in factorized form. By setting

$$\begin{cases} A = x_0^2 + \sum_{k=1}^3 \cos(2\varphi_k) x_k^2 \\ B = \sum_{k=1}^3 \sin(2\varphi_k) x_k^2 \end{cases} \tag{14.1}$$

one sees that these minors are combinations of the form $M_{ij} = P_{ij}\,A + Q_{ij}\,B$.

$$\begin{gathered} M_{12} = 2\,(\sin(\varphi_1 - \varphi_2)\,x_1\,x_2 + i\cos(\varphi_3)\,x_0\,x_3) \\ (-\cos(\varphi_1 - \varphi_2)\,(\cos(\varphi_1 - \varphi_3)\sin(\varphi_1)\,x_1^2 + \cos(\varphi_2 - \varphi_3)\sin(\varphi_2)\,x_2^2) + \\ \sin(\varphi_3)\,(\sin(\varphi_1)\sin(\varphi_2)\,x_0^2 - \cos(\varphi_1 - \varphi_3)\cos(\varphi_2 - \varphi_3)\,x_3^2)) \\ = (\sin(\varphi_1 - \varphi_2)\,x_1\,x_2 + i\cos(\varphi_3)\,x_0\,x_3) \\ (2\sin(\varphi_1)\sin(\varphi_2)\sin(\varphi_3)\,A - (\cos(\varphi_1 - \varphi_2)\cos(\varphi_3) + \sin(\varphi_1 + \varphi_2)\sin(\varphi_3))B) \end{gathered} \tag{14.2}$$

$$\begin{gathered} M_{13} = 2\,i\,(\cos(\varphi_2)\,x_0\,x_2 + i\sin(\varphi_1 - \varphi_3)\,x_1\,x_3) \\ (-\cos(\varphi_1 - \varphi_2)\,(\cos(\varphi_1 - \varphi_3)\sin(\varphi_1)\,x_1^2 + \cos(\varphi_2 - \varphi_3)\sin(\varphi_2)\,x_2^2) + \\ \sin(\varphi_3)\,(\sin(\varphi_1)\sin(\varphi_2)\,x_0^2 - \cos(\varphi_1 - \varphi_3)\cos(\varphi_2 - \varphi_3)\,x_3^2)) \\ = i\,(\cos(\varphi_2)\,x_0\,x_2 + i\sin(\varphi_1 - \varphi_3)\,x_1\,x_3) \\ (2\sin(\varphi_1)\sin(\varphi_2)\sin(\varphi_3)\,A - (\cos(\varphi_1 - \varphi_2)\cos(\varphi_3) + \sin(\varphi_1 + \varphi_2)\sin(\varphi_3))B) \end{gathered} \tag{14.3}$$

$$\begin{gathered} M_{14} = 2\,(\sin(\varphi_2)\,x_0\,x_2 + i\cos(\varphi_1 - \varphi_3)\,x_1\,x_3) \\ (\cos(\varphi_2)\,(\cos(\varphi_3)\sin(\varphi_1)\,x_0^2 + \cos(\varphi_2 - \varphi_3)\sin(\varphi_1 - \varphi_2)\,x_2^2) + \\ \sin(\varphi_1 - \varphi_3)\,(-\sin(\varphi_1)\sin(\varphi_1 - \varphi_2)\,x_1^2 + \cos(\varphi_2 - \varphi_3)\cos(\varphi_3)\,x_3^2)) \\ = (\sin(\varphi_2)\,x_0\,x_2 + i\cos(\varphi_1 - \varphi_3)\,x_1\,x_3) \\ (2\sin(\varphi_1)\cos(\varphi_2)\cos(\varphi_3)\,A - (\cos(\varphi_1)\cos(\varphi_2 - \varphi_3) - \sin(\varphi_1)\sin(\varphi_2 + \varphi_3))\,B) \end{gathered} \tag{14.4}$$

$$\begin{gathered} M_{15} = -i\cos(\varphi_1 - \varphi_2 - \varphi_3) \\ (\sin(2\,(\varphi_1 - \varphi_2))\,x_1^2\,x_2^2 + \sin(2\,(\varphi_1 - \varphi_3))\,x_1^2\,x_3^2 - x_0^2\,(\sin(2\,\varphi_2)\,x_2^2 + \sin(2\,\varphi_3)\,x_3^2)) \\ = i\cos(\varphi_1 - \varphi_2 - \varphi_3) \\ ((\sin(2\varphi_2)\,x_2^2 + \sin(2\varphi_3)\,x_3^2)\,A - (\cos(2\varphi_2)\,x_2^2 + \cos(2\varphi_3)\,x_3^2)\,B) \end{gathered} \tag{14.5}$$

$$\begin{gathered} M_{16} = -2\,(-i\cos(\varphi_1 - \varphi_2)\,x_1\,x_2 + \sin(\varphi_3)\,x_0\,x_3) \\ (\cos(\varphi_2)\,(\cos(\varphi_3)\sin(\varphi_1)\,x_0^2 + \cos(\varphi_2 - \varphi_3)\sin(\varphi_1 - \varphi_2)\,x_2^2) + \\ \sin(\varphi_1 - \varphi_3)\,(-\sin(\varphi_1)\sin(\varphi_1 - \varphi_2)\,x_1^2 + \cos(\varphi_2 - \varphi_3)\cos(\varphi_3)\,x_3^2)) \\ = -(-i\cos(\varphi_1 - \varphi_2)\,x_1\,x_2 + \sin(\varphi_3)\,x_0\,x_3) \\ (2\sin(\varphi_1)\cos(\varphi_2)\cos(\varphi_3)\,A - (\cos(\varphi_1)\cos(\varphi_2 - \varphi_3) - \sin(\varphi_1)\sin(\varphi_2 + \varphi_3))\,B) \end{gathered} \tag{14.6}$$

$$
\begin{aligned}
M_{23} &= 2\,(i\cos(\varphi_1)\,x_0\,x_1 + \sin(\varphi_2-\varphi_3)\,x_2\,x_3)\\
&(-\cos(\varphi_1-\varphi_2)\,(\cos(\varphi_1-\varphi_3)\sin(\varphi_1)\,x_1^2 + \cos(\varphi_2-\varphi_3)\sin(\varphi_2)\,x_2^2)+\\
&\sin(\varphi_3)\,(\sin(\varphi_1)\sin(\varphi_2)\,x_0^2 - \cos(\varphi_1-\varphi_3)\cos(\varphi_2-\varphi_3)\,x_3^2))\\
&= (i\cos(\varphi_1)\,x_0\,x_1 + \sin(\varphi_2-\varphi_3)\,x_2\,x_3)\\
&(2\sin(\varphi_1)\sin(\varphi_2)\sin(\varphi_3)\,A - (\cos(\varphi_1-\varphi_2)\cos(\varphi_3) + \sin(\varphi_1+\varphi_2)\sin(\varphi_3))B)
\end{aligned}
\tag{14.7}
$$

$$
\begin{aligned}
M_{24} &= 2\,(\sin(\varphi_1)\,x_0\,x_1 - i\cos(\varphi_2-\varphi_3)\,x_2\,x_3)\\
&(\cos(\varphi_1)\,(\cos(\varphi_3)\sin(\varphi_2)\,x_0^2 - \cos(\varphi_1-\varphi_3)\sin(\varphi_1-\varphi_2)\,x_1^2)+\\
&\sin(\varphi_2-\varphi_3)\,(\sin(\varphi_1-\varphi_2)\sin(\varphi_2)\,x_2^2 + \cos(\varphi_1-\varphi_3)\cos(\varphi_3)\,x_3^2))\\
&= (\sin(\varphi_1)\,x_0\,x_1 - i\cos(\varphi_2-\varphi_3)\,x_2\,x_3)\\
&(2\cos(\varphi_3)\cos(\varphi_1)\sin(\varphi_2)\,A - (\cos(\varphi_3-\varphi_1)\cos(\varphi_2) - \sin(\varphi_3+\varphi_1)\sin(\varphi_2))\,B)
\end{aligned}
\tag{14.8}
$$

$$
\begin{aligned}
M_{25} &= -2\,i\,(\cos(\varphi_1-\varphi_2)\,x_1\,x_2 - i\sin(\varphi_3)\,x_0\,x_3)\\
&(\cos(\varphi_1)\,(-\cos(\varphi_3)\sin(\varphi_2)\,x_0^2 + \cos(\varphi_1-\varphi_3)\sin(\varphi_1-\varphi_2)\,x_1^2)-\\
&\sin(\varphi_2-\varphi_3)\,(\sin(\varphi_1-\varphi_2)\sin(\varphi_2)\,x_2^2 + \cos(\varphi_1-\varphi_3)\cos(\varphi_3)\,x_3^2))\\
&= i\,(\cos(\varphi_1-\varphi_2)\,x_1\,x_2 - i\sin(\varphi_3)\,x_0\,x_3)\\
&(2\cos(\varphi_3)\cos(\varphi_1)\sin(\varphi_2)\,A - (\cos(\varphi_3-\varphi_1)\cos(\varphi_2) - \sin(\varphi_3+\varphi_1)\sin(\varphi_2))\,B)
\end{aligned}
\tag{14.9}
$$

$$
\begin{aligned}
M_{26} &= i\cos(\varphi_1-\varphi_2+\varphi_3)\\
&(\sin(2\,\varphi_1)\,x_0^2\,x_1^2 + \sin(2\,(\varphi_1-\varphi_2))\,x_1^2\,x_2^2 + (\sin(2\,\varphi_3)\,x_0^2 - \sin(2\,(\varphi_2-\varphi_3))\,x_2^2)\,x_3^2)\\
&= i\cos(\varphi_1-\varphi_2+\varphi_3)\\
&((\sin(2\varphi_1)\,x_1^2 + \sin(2\varphi_3)\,x_3^2)\,A - (\cos(2\varphi_1)\,x_1^2 + \cos(2\varphi_3)\,x_3^2)\,B)
\end{aligned}
\tag{14.10}
$$

$$
\begin{aligned}
M_{34} &= -i\cos(\varphi_1+\varphi_2-\varphi_3)\\
&(\sin(2\,\varphi_1)\,x_0^2\,x_1^2 + \sin(2\,\varphi_2)\,x_0^2\,x_2^2 + (\sin(2\,(\varphi_1-\varphi_3))\,x_1^2 + \sin(2\,(\varphi_2-\varphi_3))\,x_2^2)\,x_3^2)\\
&= -i\cos(\varphi_1+\varphi_2-\varphi_3)\\
&((\sin(2\varphi_1)\,x_1^2 + \sin(2\varphi_2)\,x_2^2)\,A - (\cos(2\varphi_1)\,x_1^2 + \cos(2\varphi_2)\,x_2^2)\,B)
\end{aligned}
\tag{14.11}
$$

$$
\begin{aligned}
M_{35} &= 2\,i\,(i\sin(\varphi_2)\,x_0\,x_2 + \cos(\varphi_1-\varphi_3)\,x_1\,x_3)\\
&(\cos(\varphi_1)\,(-\cos(\varphi_2)\sin(\varphi_3)\,x_0^2 + \cos(\varphi_1-\varphi_2)\sin(\varphi_1-\varphi_3)\,x_1^2)+\\
&\sin(\varphi_2-\varphi_3)\,(\cos(\varphi_1-\varphi_2)\cos(\varphi_2)\,x_2^2 + \sin(\varphi_1-\varphi_3)\sin(\varphi_3)\,x_3^2))\\
&= -i\,(i\sin(\varphi_2)\,x_0\,x_2 + \cos(\varphi_1-\varphi_3)\,x_1\,x_3)\\
&(2\cos(\varphi_1)\cos(\varphi_2)\sin(\varphi_3)\,A - (\cos(\varphi_1-\varphi_2)\cos(\varphi_3) - \sin(\varphi_1+\varphi_2)\sin(\varphi_3))\,B)
\end{aligned}
\tag{14.12}
$$

$$
\begin{aligned}
M_{36} &= -2\,(\sin(\varphi_1)\,x_0\,x_1 + i\cos(\varphi_2-\varphi_3)\,x_2\,x_3)\\
&(\cos(\varphi_1)\,(-\cos(\varphi_2)\sin(\varphi_3)\,x_0^2 + \cos(\varphi_1-\varphi_2)\sin(\varphi_1-\varphi_3)\,x_1^2)+\\
&\sin(\varphi_2-\varphi_3)\,(\cos(\varphi_1-\varphi_2)\cos(\varphi_2)\,x_2^2 + \sin(\varphi_1-\varphi_3)\sin(\varphi_3)\,x_3^2))\\
&= (\sin(\varphi_1)\,x_0\,x_1 + i\cos(\varphi_2-\varphi_3)\,x_2\,x_3)\\
&(2\cos(\varphi_1)\cos(\varphi_2)\sin(\varphi_3)\,A - (\cos(\varphi_1-\varphi_2)\cos(\varphi_3) - \sin(\varphi_1+\varphi_2)\sin(\varphi_3))\,B)
\end{aligned}
\tag{14.13}
$$

$$
\begin{aligned}
M_{45} &= -2\,i\,(\cos(\varphi_2)\,x_0\,x_2 - i\sin(\varphi_1-\varphi_3)\,x_1\,x_3)\\
&(\cos(\varphi_1)\,(\cos(\varphi_3)\sin(\varphi_2)\,x_0^2 - \cos(\varphi_1-\varphi_3)\sin(\varphi_1-\varphi_2)\,x_1^2)+\\
&\sin(\varphi_2-\varphi_3)\,(\sin(\varphi_1-\varphi_2)\sin(\varphi_2)\,x_2^2 + \cos(\varphi_1-\varphi_3)\cos(\varphi_3)\,x_3^2))\\
&= -i\,(\cos(\varphi_2)\,x_0\,x_2 - i\sin(\varphi_1-\varphi_3)\,x_1\,x_3)\\
&(2\cos(\varphi_3)\cos(\varphi_1)\sin(\varphi_2)\,A - (\cos(\varphi_3-\varphi_1)\cos(\varphi_2) - \sin(\varphi_3+\varphi_1)\sin(\varphi_2))\,B)
\end{aligned}
\tag{14.14}
$$

$$\begin{aligned}(14.15)\qquad & M_{46} = 2\,(-i\cos(\varphi_1)\,x_0\,x_1 + \sin(\varphi_2-\varphi_3)\,x_2\,x_3)\\ &(\cos(\varphi_2)\,(\cos(\varphi_3)\sin(\varphi_1)\,x_0^2 + \cos(\varphi_2-\varphi_3)\sin(\varphi_1-\varphi_2)\,x_2^2)+\\ &\sin(\varphi_1-\varphi_3)\,(-\sin(\varphi_1)\sin(\varphi_1-\varphi_2)\,x_1^2+\cos(\varphi_2-\varphi_3)\cos(\varphi_3)\,x_3^2))\\ &= (-i\cos(\varphi_1)\,x_0\,x_1+\sin(\varphi_2-\varphi_3)\,x_2\,x_3)\\ &(2\sin(\varphi_1)\cos(\varphi_2)\cos(\varphi_3)\,A-(\cos(\varphi_1)\cos(\varphi_2-\varphi_3)-\sin(\varphi_1)\sin(\varphi_2+\varphi_3))\,B)\end{aligned}$$

$$\begin{aligned}(14.16)\qquad & M_{56} = 2\,(\sin(\varphi_1-\varphi_2)\,x_1\,x_2 - i\cos(\varphi_3)\,x_0\,x_3)\\ &(\cos(\varphi_1)\,(\cos(\varphi_2)\sin(\varphi_3)\,x_0^2 - \cos(\varphi_1-\varphi_2)\sin(\varphi_1-\varphi_3)\,x_1^2)-\\ &\sin(\varphi_2-\varphi_3)\,(\cos(\varphi_1-\varphi_2)\cos(\varphi_2)\,x_2^2+\sin(\varphi_1-\varphi_3)\sin(\varphi_3)\,x_3^2))\\ &= (\sin(\varphi_1-\varphi_2)\,x_1\,x_2-i\cos(\varphi_3)\,x_0\,x_3)\\ &(2\cos(\varphi_1)\cos(\varphi_2)\sin(\varphi_3)\,A-(\cos(\varphi_1-\varphi_2)\cos(\varphi_3)-\sin(\varphi_1+\varphi_2)\sin(\varphi_3))\,B)\end{aligned}$$

15. Appendix 2: The sixteeen theta relations

The sixteen theta relations are the following, with

$$(w,x,y,z) = M\,(a,b,c,d)\,,$$

where

$$M := \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}$$

$$(15.1)\qquad \begin{gathered}\vartheta_2(a)\,\vartheta_2(b)\,\vartheta_2(c)\,\vartheta_2(d)+\vartheta_3(a)\,\vartheta_3(b)\,\vartheta_3(c)\,\vartheta_3(d)=\\ \vartheta_2\,(x)\,\vartheta_2\,(y)\,\vartheta_2\,(z)\,\vartheta_2\,(w)+\vartheta_3\,(x)\,\vartheta_3\,(y)\,\vartheta_3\,(z)\,\vartheta_3\,(w)\end{gathered}$$

$$(15.2)\qquad \begin{gathered}\vartheta_3(a)\,\vartheta_3(b)\,\vartheta_3(c)\,\vartheta_3(d)-\vartheta_2(a)\,\vartheta_2(b)\,\vartheta_2(c)\,\vartheta_2(d)=\\ \vartheta_1\,(x)\,\vartheta_1\,(y)\,\vartheta_1\,(z)\,\vartheta_1\,(w)+\vartheta_4\,(x)\,\vartheta_4\,(y)\,\vartheta_4\,(z)\,\vartheta_4\,(w)\end{gathered}$$

$$(15.3)\qquad \begin{gathered}\vartheta_1(a)\,\vartheta_1(b)\,\vartheta_1(c)\,\vartheta_1(d)+\vartheta_4(a)\,\vartheta_4(b)\,\vartheta_4(c)\,\vartheta_4(d)=\\ \vartheta_3\,(w)\,\vartheta_3\,(x)\,\vartheta_3\,(y)\,\vartheta_3\,(z)-\vartheta_2\,(w)\,\vartheta_2\,(x)\,\vartheta_2\,(y)\,\vartheta_2\,(z)\end{gathered}$$

$$(15.4)\qquad \begin{gathered}\vartheta_4(a)\,\vartheta_4(b)\,\vartheta_4(c)\,\vartheta_4(d)-\vartheta_1(a)\,\vartheta_1(b)\,\vartheta_1(c)\,\vartheta_1(d)=\\ \vartheta_4\,(w)\,\vartheta_4\,(x)\,\vartheta_4\,(y)\,\vartheta_4\,(z)-\vartheta_1\,(w)\,\vartheta_1\,(x)\,\vartheta_1\,(y)\,\vartheta_1\,(z)\end{gathered}$$

$$(15.5)\qquad \begin{gathered}\vartheta_1(a)\,\vartheta_1(b)\,\vartheta_2(c)\,\vartheta_2(d)+\vartheta_3(c)\,\vartheta_3(d)\,\vartheta_4(a)\,\vartheta_4(b)=\\ \vartheta_1\,(x)\,\vartheta_1\,(w)\,\vartheta_2\,(y)\,\vartheta_2\,(z)+\vartheta_3\,(y)\,\vartheta_3\,(z)\,\vartheta_4\,(x)\,\vartheta_4\,(w)\end{gathered}$$

$$(15.6)\qquad \begin{gathered}\vartheta_4(a)\,\vartheta_4(b)\,\vartheta_3(c)\,\vartheta_3(d)-\vartheta_1(a)\,\vartheta_1(b)\,\vartheta_2(c)\,\vartheta_2(d)=\\ \vartheta_1\,(y)\,\vartheta_1\,(z)\,\vartheta_2\,(x)\,\vartheta_2\,(w)+\vartheta_3\,(x)\,\vartheta_3\,(w)\,\vartheta_4\,(y)\,\vartheta_4\,(z)\end{gathered}$$

$$(15.7)\qquad \begin{gathered}\vartheta_1(a)\,\vartheta_1(b)\,\vartheta_3(c)\,\vartheta_3(d)+\vartheta_2(c)\,\vartheta_2(d)\,\vartheta_4(a)\,\vartheta_4(b)=\\ \vartheta_1\,(x)\,\vartheta_1\,(w)\,\vartheta_3\,(y)\,\vartheta_3\,(z)+\vartheta_2\,(y)\,\vartheta_2\,(z)\,\vartheta_4\,(x)\,\vartheta_4\,(w)\end{gathered}$$

$$(15.8)\qquad \begin{gathered}\vartheta_4(a)\,\vartheta_4(b)\,\vartheta_2(c)\,\vartheta_2(d)-\vartheta_1(a)\,\vartheta_1(b)\,\vartheta_3(c)\,\vartheta_3(d)=\\ \vartheta_1\,(y)\,\vartheta_1\,(z)\,\vartheta_3\,(x)\,\vartheta_3\,(w)+\vartheta_2\,(x)\,\vartheta_2\,(w)\,\vartheta_4\,(y)\,\vartheta_4\,(z)\end{gathered}$$

$$\begin{aligned}\vartheta_2(c)\,\vartheta_2(d)\,\vartheta_3(a)\,\vartheta_3(b)+\vartheta_2(a)\,\vartheta_2(b)\,\vartheta_3(c)\,\vartheta_3(d)=\\ \vartheta_2\;(x)\,\vartheta_2\;(w)\,\vartheta_3\;(y)\,\vartheta_3\;(z)+\vartheta_2\;(y)\,\vartheta_2\;(z)\,\vartheta_3\;(x)\,\vartheta_3\;(w)\end{aligned}\tag{15.9}$$

$$\begin{aligned}\vartheta_3(a)\,\vartheta_3(b)\,\vartheta_2(c)\,\vartheta_2(d)-\vartheta_2(a)\,\vartheta_2(b)\,\vartheta_3(c)\,\vartheta_3(d)=\\ \vartheta_1\;(x)\,\vartheta_1\;(w)\,\vartheta_4\;(y)\,\vartheta_4\;(z)+\vartheta_1\;(y)\,\vartheta_1\;(z)\,\vartheta_4\;(x)\,\vartheta_4\;(w)\end{aligned}\tag{15.10}$$

$$\begin{aligned}\vartheta_1(c)\,\vartheta_1(d)\,\vartheta_4(a)\,\vartheta_4(b)+\vartheta_1(a)\,\vartheta_1(b)\,\vartheta_4(c)\,\vartheta_4(d)=\\ \vartheta_3\;(w)\,\vartheta_3\;(x)\,\vartheta_2\;(y)\,\vartheta_2\;(z)-\vartheta_2\;(w)\,\vartheta_2\;(x)\,\vartheta_3\;(y)\,\vartheta_3\;(z)\end{aligned}\tag{15.11}$$

$$\begin{aligned}\vartheta_4(a)\,\vartheta_4(b)\,\vartheta_1(c)\,\vartheta_1(d)-\vartheta_1(a)\,\vartheta_1(b)\,\vartheta_4(c)\,\vartheta_4(d)=\\ \vartheta_4\;(w)\,\vartheta_4\;(x)\,\vartheta_1\;(y)\,\vartheta_1\;(z)-\vartheta_1\;(w)\,\vartheta_1\;(x)\,\vartheta_4\;(y)\,\vartheta_4\;(z)\end{aligned}\tag{15.12}$$

$$\begin{aligned}\vartheta_2(c)\,\vartheta_2(d)\,\vartheta_3(a)\,\vartheta_3(b)+\vartheta_1(c)\,\vartheta_1(d)\,\vartheta_4(a)\,\vartheta_4(b)=\\ \vartheta_2\;(y)\,\vartheta_2\;(z)\,\vartheta_3\;(x)\,\vartheta_3\;(w)+\vartheta_1\;(y)\,\vartheta_1\;(z)\,\vartheta_4\;(x)\,\vartheta_4\;(w)\end{aligned}\tag{15.13}$$

$$\begin{aligned}\vartheta_3(a)\,\vartheta_3(b)\,\vartheta_2(c)\,\vartheta_2(d)-\vartheta_4(a)\,\vartheta_4(b)\,\vartheta_1(c)\,\vartheta_1(d)=\\ \vartheta_2\;(x)\,\vartheta_2\;(w)\,\vartheta_3\;(y)\,\vartheta_3\;(z)+\vartheta_1\;(x)\,\vartheta_1\;(w)\,\vartheta_4\;(y)\,\vartheta_4\;(z)\end{aligned}\tag{15.14}$$

$$\begin{aligned}\vartheta_1(d)\,\vartheta_2(b)\,\vartheta_3(a)\,\vartheta_4(c)+\vartheta_1(c)\,\vartheta_2(a)\,\vartheta_3(b)\,\vartheta_4(d)=\\ \vartheta_1\;(w)\,\vartheta_4\;(x)\,\vartheta_2\;(y)\,\vartheta_3\;(z)-\vartheta_4\;(w)\,\vartheta_1\;(x)\,\vartheta_3\;(y)\,\vartheta_2\;(z)\end{aligned}\tag{15.15}$$

$$\begin{aligned}\vartheta_3(a)\,\vartheta_2(b)\,\vartheta_4(c)\,\vartheta_1(d)-\vartheta_2(a)\,\vartheta_3(b)\,\vartheta_1(c)\,\vartheta_4(d)=\\ \vartheta_3\;(w)\,\vartheta_2\;(x)\,\vartheta_4\;(y)\,\vartheta_1\;(z)-\vartheta_2\;(w)\,\vartheta_3\;(x)\,\vartheta_1\;(y)\,\vartheta_4\;(z)\end{aligned}\tag{15.16}$$

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